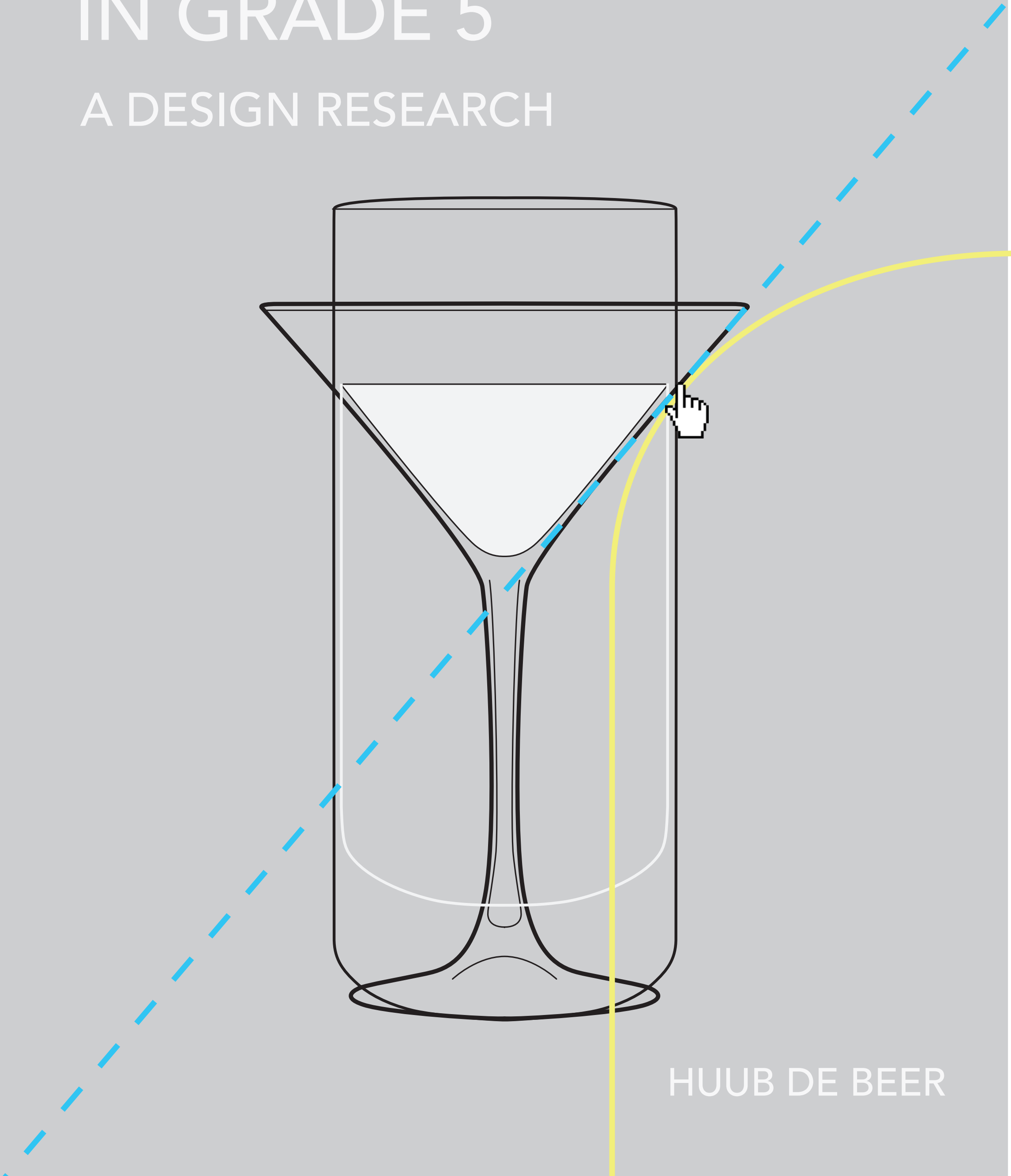
EXPLORING INSTANTANEOUS SPEED IN GRADE 5

A DESIGN RESEARCH



HUUB DE BEER

Exploring Instantaneous Speed in Grade Five
A Design Research

Huub de Beer

May 11, 2016

This research was carried out at the Eindhoven School of Eduction (ESoE), the teacher training institute of the Eindhoven University of Technology (TU/e), in the context of the Dutch Inter-university Center for Educational Research (ICO).

A catalogue record is available from the Eindhoven University of Technology Library
ISBN: 978-90-386-4038-9

Created using the free software vim, pandoc, and LaTeX
Printed by Deraumaux, Eindhoven
Cover design by Jessica Smarsch

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PROEFSCHRIFT

ter verkrijging van de graad van doctor aan de Technische Universiteit Eindhoven, op gezag van de rector magnificus prof.dr.ir. F.P.T. Baaijens, voor een commissie aangewezen door het College voor Promoties, in het openbaar te verdedigen op woensdag 11 mei 2016 om 16:00 uur

door

Huub Theodorus de Beer

geboren te Reusel

Dit proefschrift is goedgekeurd door de promotoren en de samenstelling van de promotiecommissie is als volgt:

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Het onderzoek of ontwerp dat in dit proefschrift wordt beschreven is uitgevoerd in overeenstemming met de TU/e Gedragscode Wetenschapsbeoefening.

Acknowledgements

In many ways, doing a PhD is a collaborative effort. During my PhD project I worked with primary school teachers and their students, my colleagues at the Eindhoven School of Education, and my supervisors. I am thankful for their contributions and collegiality.

I would like to thank Aniek (teaching at De Schatkist in Budel), Mayke (teaching at De Zevensprong in Eindhoven), and Erik (teaching at DOEN in Vught) for realizing my educational designs in their classrooms. Not only did they contribute their expertise as primary school teachers, they also recovered my waning faith in our education system. It is said that doing educational research is disastrous for one's faith in the education system, but you showed me that as long as there are committed teachers like you, education will be all-right.

I am grateful for the students' enthusiastic participation in the various teaching experiments. In particular, I would like to thank "Maria", "Patricia", "Roel", and "Sem" (5^{th} grade students at *De Kameleon* in 's-Hertogenbosch, using their self-chosen pseudonyms) for coming to the university on an extremely hot summer day to be my students for a day while their peers went swimming in the lake. Their contribution led me to a deeper understanding of what I was trying to do.

I would like to thank Elise, Frans, Frans, Koeno, and Michiel for their work on the STEFF project. You showed me that others were interested in and working on the same topics as I, which was a comforting thought. More so, by inviting me to join, I was able to calibrate my ideas with reference to ideas of experienced educational designers like you.

I had an interesting experience being part of the Educational Office at the ESoE. Joining Elise, Maryan, Ruurd, Veronique, Yvonne, and, later, Connie, I had the privilege to experience the master Science Education and Communication both as a student and as a member of the staff. Having graduated from that master program only a couple of years earlier, only now I came to appreciate the work and care of the ESoE staff that I took for granted when I was a student. Besides, I had a lot of fun hacking together a prototype of a student monitoring system.

Speaking of fun, I would like to thank two of my peers in particular: my roommate Ralf and busy bee Anna. Ralf, you and I were located at the far reaches of the ESoE hallway. In Summer, when most of our other colleagues had vanished, we seemed the last survivors of some apocalyptic event. I liked our bantering, absurd or otherwise. Anna, you seemed to be involved with all the extra-curricular activities going on at the ESoE, and you were somehow able to persuade me to join you in two of these activities, the colloquium committee and the ESoE newsletter. I doubt I contributed much to either, but I enjoyed working with you on these distractions.

Finally, I would like to thank my supervisors, Koeno and Michiel. Michiel's story is a sad one. He passed in the fourth year of our collaboration. I fondly remember the last day we worked together. He brought his daughter and three of her 5th grade classmates to the university to join us for a teaching experiment. In between two lessons, after lunch, we visited the amateur radio station at the university. If possible, Michiel was even more curious during the tour of the station than the children. I liked that about him. Often, during our two-weekly meetings, he would channel that curiosity to offer me an alternative perspective on my work, enabling me to look beyond the local scope of my project.

Koeno, your contribution and experience as a researcher and educational designer was invaluable. Thank you. I appreciate your extended and, after Michiel's passing, increased involvement in my project. If anything, you showed me that a passion for exploration and learning does not have to diminish over time. I think you presented me with at least one new idea each time we spoke. Our meetings became a source of inspiration to me, even though it would have sometimes been easier if we would have continued on the path we had set out on in a meeting prior. In the end, though, I felt at home with your way of supervising. You gave me the time I needed to reflect on my own learning to finish my PhD project.

Preface

At the end of 2009, I received an email inquiring if I would be interested in doing a PhD at the Eindhoven School of Education, the teacher training institute I graduated from a year prior. Attached to the message were two project proposals, one of which piqued my interest. The proposal, titled *Science & Technology Education for the Future*, outlined the need for innovative and ICT-rich science and technology education in primary school. The project aimed at developing such innovative science and technology education through a process called *design research*. I was unfamiliar with this research methodology, but after reading up on it, I found it intriguingly similar to modern software development methodologies I was familiar with.

A week before I started my PhD project, my supervisor emailed me. He explained that my project was related to another one he was involved in, also called *Science & Technology Education for the Future*, or *STEFF* for short. They were trying out an instructional sequence next week, but none of the project members were available next Thursday to observe and capture the teaching experiment on video. He asked if I was willing to step in.

That Thursday morning, on my first day as a PhD student, I stepped on the train to 's-Hertogenbosch to visit a primary school for the first time in over 15 years. A lot had changed since the last time I was in primary school. The classroom was different from how I remembered it: the blackboard was replaced with an interactive whiteboard; students sat in small groups instead of neat rows; some students walked around in class while the teacher was helping others elsewhere in the classroom. The lesson I observed was one with multiple computer activities. While the students happily moused around on their computers, the teacher and I observed them, now and then offering support to enable them to complete their tasks. I liked this hands-on approach to doing research; I went back to Eindhoven with a good feeling about the project. This would be fun!

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Introduction

Ever since the invention of the computer in the first half of the 20th century, computerization has had an increasing impact on our society. Some speak of an *information revolution*, arguing that the effects of ubiquitous computing and computer networks are fundamentally altering our self-image and the way we live our lives (Illari, 2013). Collins and Halverson (2009) compare information and communication technology (ICT) to technologies of the past that fundamentally altered our world, arguing that:

> 'No one will be able to solve complex problems or think effectively in the coming world without using digital technologies. (...) Just as reading was made necessary by the printing press and arithmetic by the introduction of money, so computer technologies are changing the very ways we think and make sense of the world.' (Collins & Halverson, 2009, p. 11)

Although the transformative powers of computer technology on our society have been recognized since the conception of the computer, that recognition was never widespread. For decades, despite their size—or perhaps precisely because of their size—, computers were almost invisible in daily life. Only after the advent of the personal computer in the 1980s and, in particular, the 1990s, the effect computers have had on society became more and more visible. While this process sped up in the 2000s, from a computer in almost every home at the start of the decade to almost everyone carrying around a personal mobile computing device at the end, society at large transformed from a (post) industrial society into an information society.

In education too, 'sophisticated computers and telecommunications are on the verge of reshaping the mission, objectives, content, and processes of schooling (...) to meet the challenge of making pupils ready for a future quite different than the immediate past.' (Dede, 2000, p. 281) In the global interconnected information society, our children have to compete with others around the world, while their skills and knowledge are becoming obsolete faster (Molnar, 1997; Gravemeijer, 2009) and more occupations are in danger of being automated (Frey & Osborne, 2013). In anticipation of these changes, ever since the late 20th century, there has been a movement to teach so-called *21st century skills*, which includes skills like problem solving, critical thinking, ICT competency, computational thinking, and innovation skills (Cobo, 2013; Rotherham & Willingham, 2010; Ferrari, Punie, & Redecker, 2012; Dede, 2010; Wing, 2010).

1.1 The need for new STEM education in primary school

Furthermore, in recognition of the historical and cultural important role that science, technology, and mathematics (STEM) play in the high-tech information society (Millar & Osborne, 1998), many a researcher voiced the need for a new STEM education in primary school (Gravemeijer, 2013; Léna, 2006; Millar & Osborne, 1998) because the reality in our primary schools does not reflect this need. STEM in primary education, particularly in the Netherlands, is often limited to a small number of hands-on activities a month (van Keulen, 2009). According to the Dutch Inspectorate for Education, only 19% of Dutch primary schools did achieve a satisfactory level of STEM education in 2009 (Inspectie van het Onderwijs, 2010, p. 45). Although this was a huge improvement over the percentage of primary schools performing satisfactory in 2004 (2%) (Inspectie van het Onderwijs, 2005), STEM in Dutch primary schools is still underdeveloped.

Since the 1960s, when STEM education meant "science for all", STEM education switched to a process-oriented approach in the 1980s and, now, to a literacy approach (Millar & Osborne, 1998). Although STEM education has become more important in the primary curriculum, its content has stayed largely the same since the 1960s, retaining

> 'its past, mid-twentieth-century emphasis, presenting science as a body

> of knowledge which is value-free, objective and detached—a succession of "facts" to be learnt, with insufficient indication of any overarching coherence and a lack of contextual relevance to the future needs of young people. The result is a growing tension between school science and contemporary science as portrayed in the media, between the needs of future specialists and the needs of young people in the workplace and as informed citizens.' (Millar & Osborne, 1998, p. 4)

Léna (2006) concurs, arguing that: 'Science curricula seem unable to convey anything but the "old" physics, and its associated, outdated representations.'(Léna, 2006, p. 5) Instead, he continues, science literacy is important for all children because it allows them to explore the world around them, to use scientific knowledge, and its results. He attributes the main problems of current STEM education in primary school to the teachers, who do not teach STEM often enough, fear its complexity, fear doing experiments, fear students' questions they cannot answer, and have a poor understanding of STEM. Because teachers do not feel safe teaching STEM, they resort to what they know from their own experiences with STEM education. Consequently, classes labeled STEM are often workshop classes (Léna, 2006; van Keulen, 2009) where pupils conduct traditional science experiments or tinker with familiar materials and tools, such as wood, nails, a hammer, cardboard, glue, and scissors.

To overcome these problems, new STEM education in primary school has been proposed (Gravemeijer, 2009; van Keulen, 2009; Millar & Osborne, 1998; Léna, 2006). This new STEM education is based on the observation that children are curious about the world around them. STEM education can build on that curiosity by giving students new means to explore their environment:

> 'It endows them with a rich understanding of our complex world, helps them practice an intelligent approach to dealing with the environment and develops their creativity and critical mind, their understanding of reality, compared to virtuality and teaches them the techniques and tools that societies have used to improve the human condition.' (Léna, 2006, p. 8)

To accomplish this goal, the following characteristics for innovative STEM education are formulated: inquiry-based, close to students' world-view, ICT-rich, and integrated into the primary curriculum. STEM education should be inquiry-based. Inquiry-based learning supports students' learning by involving them in real-life situations with an emphasis on questioning, hypothesizing, and experimenting by the students themselves (Léna, 2006; Rocard et al., 2007; J. Osborne & Dillon, 2008). At the same time STEM education should also be close to the students' world-view. Their environment has almost no relation to the STEM subculture. STEM education should try to minimize this gap between students' realities and the reality propagated by the scientific world-view (J. Osborne & Dillon, 2008). It should try to 'produce a populace who are comfortable, competent and confident with scientific and technical matters and artifacts.'(Millar & Osborne, 1998, p. 9) Furthermore, STEM education should use ICT. Not only will students grow up in a society where ICT is ubiquitous, the use of ICT has also an enormous potential in education. It enables new ways of teaching and learning (J. Osborne & Hennessy, 2003; Murphy, 2003; Bingimlas, 2009; Woodgate, Fraser, & Crellin, 2007). Finally, STEM education should be integrated into other subjects in the primary school curriculum. As it is, that curriculum is already overflowing. It will be difficult to try to fit in STEM as a separate subject. Furthermore, because STEM is an integral part of our information society, that should also be reflected in the curriculum by making it and integral part throughout the entire curriculum.

In the Netherlands, the government started the so-called VTB-Pro project in 2007, a large program to stimulate interest, knowledge, and skills of primary school teachers in the domain of science and technology. To implement this program in the South of the Netherlands, a knowledge center named KWTZ was erected by a consortium of teacher education institutes and the Eindhoven School of Education, which is part of the Eindhoven University of Technology. The main task of this knowledge center was to develop and offer in-service and pre-service teacher education. In addition, various research activities were started—among which the research that is reported in this dissertation. This thesis centers around exploring how these characteristics of innovative STEM education can be put into practice in primary school. Of course, as STEM is an extensive domain, this thesis can only focus on a small sub-domain of STEM, for which the concept of instantaneous rate of change is chosen. This choice is explained next.

1.2 Instantaneous speed in primary school?

In our information society the interpretation, representation, and manipulation of dynamic phenomena are becoming key activities. Conventionally, monitoring and controlling dynamic phenomena, in particular in real-time, was firmly rooted in the realm of STEM. With the advent of ubiquitous networked computing devices connected to sensors, however, awareness of dynamic phenomena in every-day life is growing. With smart phones, we can track our movements. With smart watches and other wearable technology we can monitor our heart rate, temperature, perspiration, and more. Government and utility companies want to introduce smart meters in every home. Combined with a smart thermostat we are able to monitor and adapt our energy usage patterns very precisely. On the horizon is the Internet of Things, promising for the future the ability to continuously monitor almost every aspect of our lives.

We monitor dynamic phenomena to gain a better understanding of them and, ultimately, to increase our control over them somehow. Key to a better understanding of dynamic phenomena is the concept of *instantaneous rate of change*. Instantaneous rate of change is defined mathematically as follows: Given a dynamic phenomenon described by a function f, the instantaneous rate of change at moment m is defined as

$$\lim_{h \to 0} \frac{f(m+h) - f(m)}{h}$$

This definition suggests that understanding of instantaneous rate of change builds on average rate of change, function, algebra, and the limit concept. In line with these suggestions, conventionally, students first encounter instantaneous speed mathematically in a calculus course, which is placed at the end of the secondary school mathematics curriculum. Due to its formal mathematical nature, however, only about 10% of all students in the USA would enroll in a calculus course (Kaput & Roschelle, 1998); in the Netherlands, the percentage of students learning calculus in high school is about three times as large[*]. Furthermore, many of those students have trouble learning calculus. According to Tall (1993), students' difficulties with calculus include the limit concept, trouble connecting calculus to real world applications, their preference for procedural knowledge over deeper mathematical understanding, and a lack of understanding of and skill with algebra and function. Clearly, calculus is unsuited for primary education.

At the same time, originating from a widely held dissatisfaction with calculus courses and the growing availability of computer technology came a push for calculus

[*] *Examenmonitor VO 2015*: In the Netherlands, calculus topics are part of "Wiskunde A" and "Wiskunde B" at the havo and vwo levels. In 2015, 73% of havo students and 49.4% of vwo students selected "Wiskunde A" while 27% of havo students and 46.6% of vwo students selected "Wiskunde B". Of all students finishing high school in 2015, 45.9% attended havo or vwo, the larger amount attending havo. The treatment of calculus topics is more complete in "Wiskunde B" than "Wiskunde A" at both levels and it is more thorough in "Wiskunde A" at the vwo level than at the havo level. The treatment of calculus in "Wiskunde B" matches that of a typical Calculus course in the USA best. This suggests that about 30% of all students in the Netherlands are taught calculus compared to the 10% in the USA as indicated by Kaput (Kaput & Roschelle, 1998).

reform (Tall, 1993; Tall, Smith, & Piez, 2008). Reform initiatives followed technological developments: the microcomputer of the late 1970s brought on approaches based on numerical algorithms, improved graphics capabilities in the workstations of the 1980s resulted in approaches based on visualization, enactive approaches followed the introduction of new interactive input devices, and once computers became powerful enough to run computer algebra systems approaches based on those gained traction as well (Tall, 1997; Tall et al., 2008). Although most calculus reform initiatives focused either on improving traditional calculus courses or changing the mathematics curriculum in service of teaching and learning calculus in the formal mathematical sense (Tall, 1997; Tall et al., 2008), other researchers went beyond the educational implications outlined by traditional calculus.

For example, a group of researchers led by James Kaput saw in ICT an opportunity to democratize access to calculus (Kaput, 1994; Kaput, 1997; Kaput & Roschelle, 1998): with the advent of affordable computer technology, every child could and should be enabled to learn the mathematics of change and variation (Roschelle, Kaput, & Stroup, 2000). Therefore, in 1993, the SimCalc project was started. After showing that the average student could indeed successfully learn the mathematics of change and variation during the try-out phase, the SimCalc project was scaled up to find out how to integrate it into the curriculum of middle school and high school (Kaput & Schorr, 2007). Beyond this 'Kaputian program', as Tall (2013) called it, other researchers too focused on learning of calculus-like concepts already in elementary or middle school (Nemirovsky, 1993; Thompson, 1994b; Boyd & Rubin, 1996; Nemirovsky, Tierney, & Wright, 1998; Noble, Nemirovsky, Wright, & Tierney, 2001; Stroup, 2002; Ebersbach & Wilkening, 2007; van Galen & Gravemeijer, 2010). These initiatives suggest that calculus-like topics can be suited for primary education after all.

In a way, the computerization of our society led to a convergence of a need for a better understanding of instantaneous rate of change and new possibilities to teaching and learning calculus-like topics offered by computer technology. Because primary school students will not have developed the level of abstraction associated with the use of "rate of change" in the literature, the term "speed" is preferred. Currently, instantaneous speed is not part of the primary school curriculum. To explore teaching topics in earlier grade levels than they are usually taught, design research is well-suited (Kelly, 2013). Therefore, to explore the coming together of the need to better understand instantaneous speed from an early age on and the possibilities for innovative ways of learning offered by computer technology, a design research project was started on:

How can we teach instantaneous speed in grade five?

1.3 Design research

In the Netherlands ideas similar to design research have been explored since the 1970s by Freudenthal, Streefland, and Gravemeijer (Gravemeijer & Cobb, 2013; van Eerde, 2013). Internationally, however, design research gained traction in the 1990s—many point to the seminal work of Brown (1992) and Collins (1992) as the starting point of this development—as a reaction to a perceived gap between educational practice and research (The Design-Based Research Collective, 2003; van Eerde, 2013; Reeves, McKenney, & Herrington, 2010). To bridge this gap, design research seeks to address:

- 'The need to address theoretical questions about the nature of learning in context.
- The need for approaches to the study of learning phenomena in the real world rather than the laboratory.
- The need to go beyond narrow measures of learning.
- The need to derive research findings from formative evaluation.' (Collins, Joseph, & Bielaczyc, 2004, p. 16)

In addressing these needs, design research builds on educational design. It uses an interventionist process of iterative refinement to create some instructional artifact that takes real-world classroom practice into account (Cobb, 2003; The Design-Based Research Collective, 2003; Barab & Squire, 2004; Reimann, 2011; Plomp, 2013; Bakker & van Eerde, 2013). In contrast to educational design, however, which is product-driven, design research aims at developing theory where 'the design is conceived not just to meet local needs, but to advance a theoretical agenda, to uncover, explore, and confirm theoretical relationships.' (Barab & Squire, 2004, p. 5)

Plomp (2013) distinguishes two main categories of design research, each with a different aim (Plomp, 2013): "developmental studies" aim at developing research-based design theories for use in educational design projects, while "validation studies" aim at exploring innovative learning ecologies to develop a local instruction theory (LIT). The latter category fits the design research project described in this thesis. In particular, the design research approach outlined in Gravemeijer and Cobb (2013)[†] is applied to answer the research question, *how can we teach instantaneous speed in*

[†]Gravemeijer and Cobb (2013) is an updated version of Gravemeijer and Cobb (2006).

grade five, by developing a local instruction theory on teaching instantaneous speed in 5th grade.

A LIT pays attention to the intended learning goals, the instructional starting points implied by students' prior instruction and their intuitive understanding, and it

> 'consists of conjectures about a possible learning process, together with conjectures about possible means of supporting that learning process. The means of support encompass potentially productive instructional activities and (computer) tools as well as an envisioned classroom culture and the proactive role of the teacher.' (Gravemeijer & Cobb, 2013, p. 78)

To start off the design research process, an initial LIT is formulated based on the literature and any other source that might contribute to the researcher's understanding of students' prior conceptions (Figure 1.1, p. 9, starting up). After this starting-up phase, the LIT is elaborated, adapted, and refined in one or more design experiments (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) or macro design-cycles (Figure 1.1; three design experiments were performed). A design experiment consists of three phases. First, in the preparatory phase, the LIT is (re)formulated and, based on that LIT, an instructional design is developed. Then, in the second phase, that design is tried in a teaching experiment consisting of a sequence of micro design-cycles of redeveloping, testing, and evaluating instructional activities and materials. Finally, in the retrospective analysis phase, an analysis of what happened during the teaching experiments informs the refinement of the design, and an analysis why this happened informs the refinement of the LIT. This LIT is the starting point of the next design experiment.

This design research process yields both a working prototype and a theory on how that prototype works. In line with the theory-driven nature of design research the focus in this thesis is on the LIT rather than on the prototype. Nevertheless, parts of the prototype and its development will be discussed while elaborating on the development of the LIT. For a more thorough exploration of the prototype, in particular the educational simulation software that was developed and tested in the various teaching experiments, the website https://heerdebeer.org/DR/ is set up.

1.4 Overview of the design research project

Before detailing the organization and content of this dissertation, an overview is given in terms of the four partial projects which together comprise the design research project (see Figure 1.1): after the starting-up phase, three subsequent design experiments

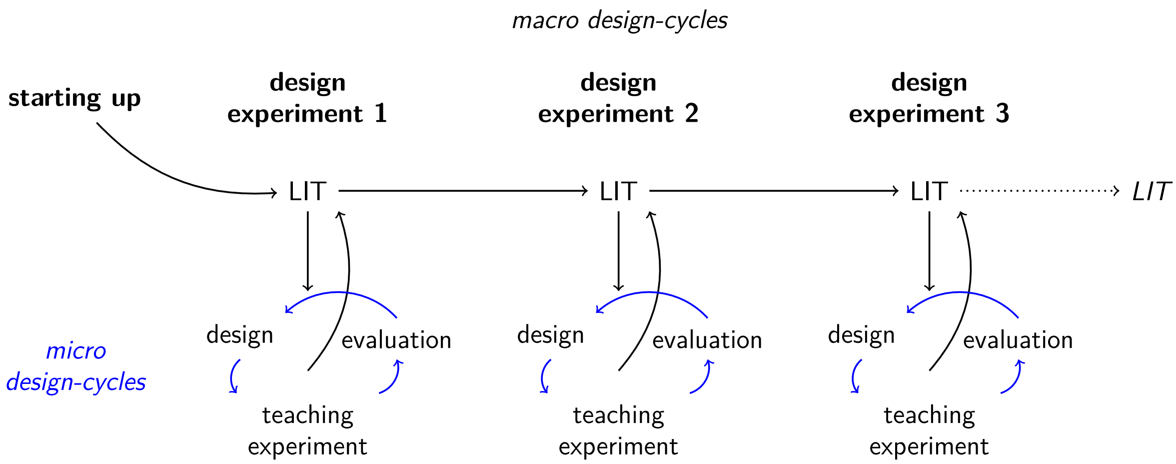


Figure 1.1: Overview of the design research project: after a starting-up phase three successive design experiments were conducted. (Taken from de Beer, Gravemeijer, and van Eijck (in preparation-b))

followed. Given the iterative nature of design research, the experiences gained and results found in earlier design experiments feed into the next. As result of this iterative and cumulative character, the findings of the starting-up phase and the first two design experiments are reflected and apparent in design experiment 3. In this thesis most attention is therefore paid to that last design experiment (Chapter 3) and the substantive relationships between that design experiment and the three partial projects that preceded it (Chapters 4 and 5). In addition, when construing the proposed LIT in Chapter 5 the data of the earlier design experiments are included in order to create a broader empirical basis. To offer a sound background for those data an overview will be given of all four partial projects in terms of their research context, data collection, and procedure of analyzing that collected data.

Because the starting-up phase is discussed in detail in Chapter 2, this section focuses on the three design experiments. Each design experiment started by designing an instructional sequence based on the ideas put forward in the LIT. Subsequently, that instructional sequence was tried in a teaching experiment while a multitude of data was collected. Before elaborating on the procedure of analysis used during the design experiments, the research context of all four partial projects is sketched using Table 1.2 detailing the background of the participants and Table 1.1 detailing the origin and type of the data collected.

	video/audio	transcript	computer session	student products	observations
Starting up					
one-on-one teaching experiments	×	×	×	×	[1]
Design experiment 1					
lesson 1	×	×		×	×
lesson 2	×	×		×	×
lesson 3	×	×		×	×
lesson 4	×	×	×	×	×
one-on-one teaching experiments	×	×	×	×	[1]
lesson 5	×	×		×	×
Design experiment 2 (at university)					
lesson 1	×	×	×	×	[1]
lesson 2	×	×	×	×	[1]
lesson 3	×	×	×	×	[1]
Design experiment 3 (2 classrooms)					
lesson 1	×	×	×[2]	×	×
lesson 2	×	×	×[2]	×	×
lesson 3	×	×	×[2]	×	×
lesson 4	×	×	×[2]	×	×
evaluation by students				×	

[1]: when the author acted as a teacher in the one-on-one teaching experiments and during design experiment 2, no observations were made.
[2]: only the teacher's computer session was captured during the lessons.

Table 1.1: Data collected during the four partial projects of the design research project. Besides the data collected during the lessons and the one-on-one teaching experiments, the meetings with the teacher in design experiments 1 and 3 to prepare and evaluate the lessons were captured on audio or video as well. This data has not been used in this dissertation, however.

	classroom with teacher	students	gifted students	5^{th} grade	mixed $5^{th}/6^{th}$ grade	mixed $4^{th}/5^{th}/6^{th}$ grade	boys	girls
starting up		9		×			4	5
design experiment 1	×	25	21		×		15	10
design experiment 2		4		×			2	2
design experiment 3, classroom C1	×	24	24			×	17	7
design experiment 3, classroom C2	×	24	24			×	18	6
total		86	69				56	30

Table 1.2: Participants in the four partial projects during the design research project.

1.4.1 Research context and data collection

Starting up

While starting up the design research project, it became apparent that a literature review did not offer enough support to formulate an initial LIT. In addition, 8 one-on-one teaching experiments (Steffe & Thompson, 2000) were performed with 9[‡] 5^{th} grade students to explore their prior understanding of speed in situations with two co-varying quantities using computer simulations and Cartesian graphs. Spread over three separate days, these teaching experiments were performed during school hours at the students' school; one of their teachers was present also and supported the researcher in having the students think aloud. This teacher discussed the experiments with the researcher before and after they took place.

These meetings with the teacher, as well as the experiments were captured on video. The video recordings of the teaching experiments were transcribed. During the experiments, the computer session was captured as well, from which the student products were collected.

[‡]In the third one-on-one teaching experiment, a pair of students participated.

Design experiment 1

What was learned during the starting up phase allowed for the formulation of the initial LIT, which, together with the experiences gained during the starting-up phase of the design research project formed the basis for the development of a five-lesson instructional sequence on instantaneous speed. This instructional sequence was tested in a gifted mixed 5th/6th grade classroom with 21 students and their experienced teacher. During the test, 4 students from the regular mixed 5th/6th grade classroom participated as well. These four students performed above average in mathematics and were regularly invited to participate in mathematical activities in the gifted classroom. There were 15 boys and 10 girls.

The five lessons in the instructional sequence were each spaced by four through seven days. Not all students took part in all five lessons due to illness, standard test taking, and other activities. Over all, however, there were always more than 15 students in class during the lessons. Two days before the last lesson, three one-on-one teaching experiments were performed with three pairs of students: one pair from the regular 5th/6th grade and two pairs from the gifted program. The intent was to have a preparatory and evaluational meeting with the teacher before and after each lesson. Unfortunately, there were no such meetings before the 2nd and 5th lesson, nor a separate meeting to evaluate the 4th. Nevertheless, each lesson was discussed briefly during and at the end of each lesson.

The teacher did have many years of experience teaching and ever since the introduction of the gifted program three years prior, she had been teaching the gifted 5th/6th grade. In the gifted program, students spend less time on the core curriculum, making time for other subjects and more challenging projects. The gifted program was set up to pay attention to the special needs of gifted children. In the program, students work on their social-emotional development, learn meta-cognitive skills, and improve their overall happiness about going to school. Nevertheless, the class behaved like any other class.

A multitude of data was collected during the teaching experiments (see Table 1.1). All lessons and meetings with the teacher were captured on video, of which the class discussions were transcribed. During the lessons the students worked on tasks via web-based interactive worksheets. Their answers were gathered, tabulated, and used to enrich the transcripts. During the lessons, the researcher present made observations. At the end of the 5th lesson, the students were given an open ended evaluation worksheet, which was filled in when the researcher was not present. The one-on-one teaching experiments were also captured on video, the computer sessions recorded, and the students' work on paper collected.

Design experiment 2

After reflecting on the results of design experiment 1, it became clear that the LIT had to be revised significantly. To try out these new ideas, a three-lesson instructional sequence was developed and tried in a small-scale teaching experiment. Four above average performing 5th grade students (2 boys and 2 girls) were invited to visit the university and participate in the teaching experiment. The three lessons were split up in two sessions, one before and one after lunch. The author acted as teacher.

During the teaching experiment, there were two kind of activities: the students worked in pairs on an assignment or the teacher led a discussion with all 4 students. These discussions as well as the students' work in pairs was recorded on video, all of which was transcribed later. The computer sessions of both pairs and the teacher were captured as well. Students' products on paper worksheets were collected. Following all 4 students closely during the teaching experiment allowed for developing a deeper understanding of the LIT in terms of students' reasoning. In this sense, design experiment 2 acted as a try-out for the classroom teaching experiments in design experiment 3.

Design experiment 3

In design experiment 3 a four-lesson instructional sequence was developed and tried at a school for gifted children. Twice a week, selected students would come from all over the municipality to attend the program for half a day during normal school hours. The gifted program focused on students' creativity and social-emotional development whilst offering an intellectual challenging environment with topics in the area of language and culture.

Two classes of 24 students from grades 4-6 participated in the teaching experiments. Classroom C1 (17 boys and 7 girls) participated on four Friday afternoons and classroom C2 (18 boys and 6 girls) participated on four Monday afternoons. Although both classrooms were taught by the same pair of teachers, only one teacher participated in our study. He was a novice teacher with less than three years of experience who also worked as a project manager and teacher of "media literacy" at the teacher training institute of a nearby college. He confessed to an affinity for science and technology.

Before the first lesson, the researcher discussed the whole instructional sequence with the teacher. Before and after each lesson, the teacher and researcher discussed the lesson and prepared for the next. These sessions were recorded. While the lessons were recorded on video, the researcher made observations and captured the teacher's computer session. Each lesson the students' products were collected as well.

1.4.2 Procedure of analysis

The procedure of analysis used during each of the three design experiments was a two-step retrospective analysis modeled after Glaser and Strauss's (1967) comparative method. In particular, the elaboration of Cobb and Whitenack (1996) on this method was used. After formulating conjectures about *What happened?* during the teaching experiments and testing these conjectures against the data collected, a second round of analysis was carried out by formulating conjectures about *Why did it happen?*. These conjectures were also tested against the data collection.

Although all data sources contributed to develop a better understanding of students' learning processes, the transcriptions of whole-class discussions and student products in particular formed the basis for the two rounds of formulating and testing conjectures during the retrospective analysis. The conjectures that validated the ideas put forth in the LIT in terms of what happened and why it did happen were carried over to the LIT of the next design experiment. The conjectures that were refuted were taken as indications of a mismatch between the researchers' ideas about the students' learning processes and their actual learning processes. In these cases, to improve the LIT, it was tried to generate new explanatory conjectures through a process of abductive reasoning (Chapter 3). These new conjectures were added to the LIT of the next design experiment. Thus refining the LIT.

1.5 Organization and content of this thesis

Again, it is emphasized that design research is a process of iterative refinement: each design experiment builds directly on the results found and experiences gained in previous design experiments and the starting-up phase. This thesis, however, is not set up as a monograph offering a continuous narrative detailing the design research project. Instead a format is chosen that consists of four separate studies that did emerge from the design research project. Each study addresses a different topic and has been developed into a self-contained article. These articles are included as such in Chapters 2–5 in this thesis. As a result, however, some repetitions in these chapters are unavoidable.

This dissertation brings together the following four studies that emerged from the design research project:

1. *The first study (Chapter 2) deals with finding starting points for the initial LIT.* It starts with a review of the literature on primary school students' conceptions of speed. However, instantaneous characteristics of speed are not part of the primary school curriculum and, therefore, are not covered in this literature. Because instantaneous speed is conventionally treated first in a calculus course, the literature review is extended towards the literature on teaching calculus-like topics early on in the mathematics curriculum. Although this review did not allow for an initial LIT on instantaneous speed to be formulated, it offered enough pointers to perform a small-scale study to explore 5th graders' conceptions of speed. After presenting the design, conduction, and analysis of 8 one-on-one teaching experiments about speed in the context of filling glassware, the results lead to the formulation of starting points of the initial LIT for teaching instantaneous speed in grade five.

2. *The second study (Chapter 3) focuses on the generation of new theoretical claims in design research by means of abductive reasoning.* This process is showcased by describing the introduction of a new explanatory conjecture during the retrospective analysis of design experiment 3. To that end, the LIT and the instructional sequence instantiated by it are described, as are the teaching experiments leading up to the surprising event that triggered the abductive process: in one of the two participating classrooms wherein the instructional sequence was tried, the students themselves invented the curve as a Cartesian graph while in the other classroom the teacher had to introduce the curve. Then, by formulating conjectures about what happened during the teaching experiments and grounding these in the data collected, a second round of analysis is performed to formulate and ground a conjecture about why the students in one classroom were able to invent the curve while the others were not. This study ends with a discussion about generating new theory in design research and the generalizability of such theory.

3. *The third study (Chapter 4) is an effort to contribute to the codification of educational design practices.* Indeed, design research builds on educational design. In the followed design research approach of Gravemeijer and Cobb (2013), the research part heavily leans on the virtual replicability of design experiments. This asks for a detailed account of the design research process and the researchers' own learning process embedded in it. In this sense, design research may offer insight on how educational designers might document their practices and knowledge. In this study, the design research process is taken

as an example to illuminate the researchers' learning process by tracking in detail the development of the instructional sequence from the starting-up phase through the three subsequent design experiments. After summarizing the yield of the researchers' own learning process in terms of a prototypical instructional sequence on instantaneous speed and the emerging LIT, the justification of the theoretical claims is discussed with respect to their origin and development. Furthermore, the utility of the LIT is discussed in light of the difference between the communities of design research and educational design.

4. *The fourth study (Chapter 5) is dedicated to the temporary endpoint and the cumulative yield of the iteration of design experiments, both in terms of insights in student thinking and in terms of a proposed LIT on teaching instantaneous speed in 5th grade.* To formulate that proposed LIT the design and theoretical underpinnings of the local instruction theory are elaborated in terms of the theory of realistic mathematics education. From a methodological perspective the argumentative grammar suggested by Cobb, Jackson, and Munoz (in press) is followed. According to Cobb et al. (in press), the justification of the theoretical findings of a design experiment follows from a) showing that the students' learning process is due to their participation in the design experiment, b) documenting that learning process, and c) enumerating the necessary means of support for that learning process to occur. Instead of documenting the learning process of the students in one go, however, first the key learning moments are described that came to the fore as patterns in the data of all teaching experiments. Then, the proposed LIT that can be construed on the basis of those learning moments is formulated by describing the students' learning processes and the necessary means of support to bring these learning processes to fruit. Finally, the proposed LIT is placed in the literature and discussed regarding further avenues of research.

Finally, in Chapter 6, a summary of this thesis is given, followed by discussion of the proposed LIT. After evaluating the proposed LIT in light of the characteristics for new STEM education put forth in this Introduction, possible avenues for adaptation and further research on teaching instantaneous speed in 5th grade are explored. In particular, the design research methodology delineated in this thesis is discussed before envisioning a potential design research project that would disseminate the proposed LIT to a larger audience, including real-world practice.

2

Investigating 5th grade students' level of covariational reasoning[*]

2.1 Introduction

In order to find instructional starting points for a 5th grade course on instantaneous speed, we designed and performed 8 one-on-one teaching experiments (Steffe & Thompson, 2000) in the context of filling glassware with nine 5th grade students[†]. In designing and executing the experiments, we drew on Carlson et. al.'s (2002) covariation framework of levels of reasoning about two co-varying quantities. The covariation framework is originally designed for college students. However, when preparing for a design experiment on teaching instantaneous speed to 5th grade students, we needed to establish potential starting points for the students. We decided to try to find out if the covariation framework could help us to get a handle on 5th grade students' covariational reasoning. We realized that applying a framework designed for

[*]This chapter is based on Beer, H. de, Gravemeijer, K., & Eijck, M. van. (submitted) Investigating 5th grade students' level of covariational reasoning.

[†]In the third one-on-one teaching experiment, a pair of students participated

older students demanded for a careful interpretation of the results. We will therefore consider the results in the context of our overall aim to obtain information that can help us design a course on instantaneous speed for 5th grade students. To elucidate this goal we will first discuss the broader background of the encompassing project to situate the research and explicate our interest in 5th grade students' covariational reasoning. We start by briefly explaining why we think it is important to teach this topic. Next, we discuss the literature on speed at the primary school level, and on teaching calculus-like topics (such as instantaneous speed) early in the curriculum

2.1.1 The need for understanding dynamic phenomena

The interpretation, representation, and manipulation of dynamic phenomena play an increasing important role in our high-tech society. One of the core concepts in understanding these phenomena is instantaneous rate of change. To participate successfully in our future society, one needs a sound mathematical understanding of rate of change. This suggests adding this topic to the primary school curriculum. Although we prefer to use the term "speed" instead of "rate of change" in the context of primary education, as primary school students will not have developed the level of abstraction that is associated with the use of "rate of change" in the literature.

To the extent that speed is addressed in primary education, it is mainly treated as a (constant) ratio or an average speed. Instantaneous speed is not part of the primary curriculum. There have been studies on young students' qualitative understanding of continuous processes of change (Nemirovsky, 1993; Boyd & Rubin, 1996; Nemirovsky et al., 1998; Stroup, 2002). However, traditionally, instantaneous characteristics of speed are first treated mathematically in calculus courses. Consequently, most literature on teaching and learning instantaneous rate of change concerns secondary and higher education. Although this literature may provide some valuable insights, it is not (directly) applicable to primary education. To fill this gap, we started a design research project on teaching instantaneous speed in primary school. Following Kaput and others (Thompson, 1994b; Kaput & Schorr, 2007; Stroup, 2002) we decided to use computer simulations and graphs to enable students without much formal mathematical baggage to explore and reason about change.

We chose the context of filling glassware because of students' familiarity with filling glasses and its allowance for non-linear situations. The origin of this context can be traced back to Swan (1985), who used it in a secondary text book on functions and graphs. Swan also offered a "microcomputer program" called *BOTTLES* that simulated filling a round-bottom flask while simultaneously drawing the graph alongside the glass. Since then this "bottle problem" has been used by many others, both in middle school (McCoy, Barger, Barnett, & Combs, 2012) and primary school textbooks[‡], while Carlson et al. (2002) used it to analyze students' covariational reasoning. We add to this literature by using this type of problems to explore primary school students' emerging intuitive understanding of speed. Therefore, to get a better sense of students' prior understanding of (instantaneous) speed in situations with two co-varying quantities than offered by the literature, we performed a one-on-one teaching experiment (Steffe & Thompson, 2000) with nine 5th graders.

2.1.2 Conceptions of speed

The study of children's conceptions of speed can be traced back to the work of Piaget (Piaget, 1970; Stroup, 2002; Wilkening & Huber, 2002). Piaget found that children develop concepts of speed and distance before that of time (Wilkening & Huber, 2002); children derive their concept of time from proportional correspondence between distance and time (Thompson, 1994b). They have trouble relating distance and duration of movement, and relating both correctly to speed (Groves & Doig, 2003). For example, without regard for distance, they erroneously equate the shortest duration with the fastest movement: "Shortest equals fastest" (Groves & Doig, 2003, p. 80). Thompson (1994b) studied a 5th grade student struggling to extend a conception of average speed towards a more general conception of rate. He found that the student initially interpreted speed as a distance, and time as a ratio; the student talked in terms of speed-lengths. Thompson observed that the conventional instruction of speed as distance-over-time, would not have worked in this case: the student came to understand speed as a rate only after understanding it as a (constant) ratio first (Thompson, 1994b). Stroup (2002), too, critiqued conventional instruction of speed for it valued intuitive understanding only as a transitional phase towards a ratio-based understanding of rate. Instead, he argued that developing a qualitative understanding of rate was a worthwhile enterprise in itself.

Stroup (2002) worked this qualitative understanding of rate into an approach to teaching calculus related concepts, called "qualitative calculus". Building on earlier work by Confrey and Smith (1994) on young students' intuitive understanding of rate in exponential situations, Stroup's qualitative calculus starts with students exploring situations with varying rate instead of (linear) situations with constant rate, as was common in conventional approaches. Confrey and Smith (1994) found that, when

[‡]Some Dutch arithmetic text books for primary school offer activities involving bottles and iconic graphs where students have to match four different bottles with the corresponding graphs. However, those activities seem to be isolated instances of the use of curves and are not connected to the concept of speed.

using graphs, students would connect a graph's slope with the corresponding rate of change; this understanding was 'more holistic than the analytic "unit per unit"' (Confrey & Smith, 1994, p. 157) or ratio-based understanding. Children appear to have an intuitive understanding of non-linear functions before those are taught formally (Ebersbach & Wilkening, 2007; Confrey & Smith, 1994).

Save for those studies on students' conceptions of varying rates, almost all research on children's conceptions of rate has focused on situations with constant or average rate of change and on finished processes of change. As a result, instantaneous characteristics have hardly been addressed in the literature. Traditionally, students explore these instantaneous aspects of change for the first time mathematically in a calculus course. Therefore, we looked at the literature on teaching calculus-like topics early in the mathematics curriculum to find possible instructional starting points for teaching instantaneous speed in primary school.

2.1.3 Teaching calculus-like topics early

Research on teaching calculus-like topics early in the mathematics curriculum is part of a longer tradition of calculus reform that originated from a widely held dissatisfaction with traditional calculus courses (Tall et al., 2008) and the advent of affordable computer technology. Tall (2010) characterized most of these reform approaches as 'largely a retention of traditional calculus ideas now supported by dynamic graphics for illustration and symbolic manipulation for computation.' (p. 3) Some researchers, however, went beyond conventional calculus and focused on learning calculus-like concepts already in primary school (Nemirovsky, 1993; Thompson, 1994b; Boyd & Rubin, 1996; Nemirovsky et al., 1998; Noble et al., 2001; Stroup, 2002; Ebersbach & Wilkening, 2007; van Galen & Gravemeijer, 2010; Kaput & Schorr, 2007). Among these initiatives, two characteristics are invariant: simulations and Cartesian graphs.

Of the "big three" representations of calculus, graphical, numerical, and symbolical (Kaput, 1998), graphs seem best suited for supporting younger students on exploring dynamic phenomena. It allows access to a dynamic situation as a whole, and not just a set of data points. Graphing is a marginal topic in primary school, however (Leinhardt, Zaslavsky, & Stein, 1990), and, for students in middle school and up, graphing appears far from trivial (Leinhardt et al., 1990). Research of Mevarech and Kramarsky (1997) showed that student could overcome superficial graphing errors during a basic graphing course, but more fundamental alternative conceptions remained.

Roth and McGinn (1997) attributes the abundance of difficulties reported in the literature to a tendency to look at students' graphing misconceptions which, given students' inexperience with graphing, are found aplenty. As an alternative, they plead for a social-cultural approach that allows students to become practitioners of graphing. In a similar vein, diSessa (1991) proposes that students invent representations of motion instead of being offered ready-made representations such as Cartesian graphs. Sixth grade students already dispose of a lot of "meta-representational competency" (diSessa & Sherin, 2000), such as being able to discuss relevant aspects of the way the situation is represented and to improve and reuse notations they developed earlier (diSessa, Hammer, Sherin, & Kolpakowski, 1991).

In addition to this reinventing approach, remarkable graphing skills are reported about primary school students who are supported by information and communication technology (ICT) (Phillips, 1997; Ainley, Nardi, & Pratt, 2000; van den Berg, Schweickert, & Manneveld, 2009). Ainley (1995) studied 4th graders exploring their own growth using spreadsheets and graphs generated from these spreadsheets. They found that "active graphing", i.e., using graphs as exploratory devices (Pratt, 1995; Ainley et al., 2000), allowed students to successfully use graphs. Similarly, primary school students appeared to perform well on graphing tasks when using sensor-based graphs (van den Berg et al., 2009). There are several explanations offered for this success. Roth and McGinn (1997) argue that this success probably lies in the role of graphs both as artifacts to reason with, and as ways to talk about the given situation. Computer generated graphs allow students not only to explore dynamic situations without the need to know graphing conventions, but in doing so, they also learn these conventions (Ainley et al., 2000). Furthermore, students focus less on discrete data points than when plotting graphs manually (Barton, 1997), allowing them to see the shape of the relationship between two co-varying variables (Ainley et al., 2000). Finally, when drawing graphs by hand, students spend less time actually using graphs (Barton, 1997). On the other hand, van Galen, Gravemeijer, van Mulken, and Quant (2012) advocate having students draw graphs manually more often to experience and adjust their possible alternative conceptions of graphing conventions.

Between the initiatives to teach calculus-like topics in primary school, graphs are almost always accompanied by and connected with computer simulations. Simulations seem a natural fit (Tversky, 2002) for calculus education because calculus was invented to describe dynamic phenomena. ICT enables students to solve more meaningful, complex, and realistic problems (Ainley, Pratt, & Nardi, 2001). It limits the cognitive load, enabling students to reason about complex problems (Schnotz & Rasch, 2005, 3). However, if these simulations lower the mental effort of students, the learning gains might be limited (Schnotz & Rasch, 2005, 3). Computer simulations can function as black boxes for students, enabling them to study the results of change without needing a deeper understanding of the mathematics of change. Gravemeijer, Cobb, Bowers, and Whitenack (2000) argue that linking reality to mathematics through ICT is not enough to induce adequate learning processes, for that a suitable instructional sequence is needed. Students need to learn how to interpret and use simulations before simulations can support learning (Ploetzner, Lippitsch, Galmbacher, Heuer, & Scherrer, 2009). Nonetheless, simulations can enable inquiry-based learning approaches because it allows students to explore phenomena that they otherwise cannot explore (Chang, 2012).

In conclusion, research suggest young students are able to explore rate of change mathematically using computer simulations when these simulations are embedded in a suitable instructional sequence. Students appear to be able to express their understanding through graphs, even when they lack graphing experience, if they are sufficiently supported in using graphs.

2.2 *Establishing 5th grade students' level of covariational reasoning*

2.2.1 *Theoretical background of the covariation framework*

In preparing for the study on teaching instantaneous speed to 5th graders, we needed to find out what to expect the students' starting position to be. More specifically, we were interested in their level of covariational reasoning in relation to graphing. As this was exactly the theme that Carlson et al. (2002) had been investigating with older students, we wondered if we could use the covariation framework they developed, to determine 5th grade students' level of reasoning. Carlson et al. (2002) define covariational reasoning as the 'cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other' (Carlson et al., 2002, p. 354). In relation to this, they refer to Thompson's conception of "image", which he describes as 'dynamic, originating in bodily actions and movements of attention, and as the source and carrier of mental operations' (Thompson, 1994a, p. 231, as cited by Carlson et. al. (2002)). The dynamic character of an image of covariation is mentioned by many researchers. This dynamic notion may, for instance, be envisioned as a point moving along a graph. Covariation is almost invariably linked to graphs, which is probably linked to the conception of a function as a set of number pairs (Sfard, 1991).

The covariational framework described by Carlson et al. (2002) consists of five developmental levels of covariational reasoning that were identified through analyzing the behavior of students solving problems about two co-varying quantities (Carlson et al., 2002).

The first level indicates the understanding that two quantities change in relation to each other. At the second level one understands that relation in terms of the direction of change of one quantity given increase in the other. At level three, understanding also encompasses the amount of change of one quantity given the amount of change of the other. Understanding at the fourth and fifth level concerns, respectively, average rate of change of a quantity given uniform increases of the other quantity and the instantaneous rate of change of a quantity given continuous change of the other quantity. A student reasons on level three, four, or five if he or she exhibits the behavior corresponding to that level and, at least, exhibits behavior corresponding to the more basic levels one and two (Carlson, Oehrtman, & Engelke, 2010).

Carlson et al. (2002) enumerate these behaviors in terms of students' graphing and accompanying verbal expressions (paraphrased in the context of filling glassware from (Carlson et al., 2002, Table 1 on p. 357)): Given a situation with co-varying quantities water level height and volume,

level 1 Students express an understanding of the two quantities involved by labeling the axes with (water level) height and volume.

level 2 Students draw a straight line to indicate that the water level height rises while the volume grows.

level 3 Students focus on the amount that the water level rises when the volume grows by plotting certain points or draw straight line segments between these points.

level 4 Students draw a continuous graph consisting of straight line segments; they understand the different rates of change of the water level height on the subsequent uniform intervals of volume growth.

level 5 Students draw a curve to express an understanding that the water level height changes continuously—thus at every moment—while the volume grows.

Carlson et al. (2002) warn for so called pseudo behavior (Vinner, 1997) as students may display behavior of a certain level without actually reasoning on that level. For example, a student could draw a curve to describe some dynamic phenomenon—which would indicate level 5 reasoning—, not because she understands what a curve means, but because she has seen these kind of graphs being used in this situation before. Therefore, to determine a students' level of covariational reasoning, multiple supporting indications of reasoning at that level are needed.

2.2.2 Research question

In light of our interest in potential starting points for an instructional sequence on instantaneous speed in 5th grade in the context of filling glassware, we want to answer the following research question:

How and at what level of the covariation framework come 5th graders to reason about two co-varying quantities in the context of filling glassware?

2.2.3 Research method

We designed a one-on-one teaching experiment (Steffe & Thompson, 2000), in which we integrated computer simulations and graphing, to explore 5th graders' initial understanding of situations with co-varying quantities and speed.

The idea of a one-on-one teaching experiment is to engage one student in some instructional activity and expand that activity to capture the student's learning process as good as possible. We designed an interactive simulation of filling glassware to enable primary school students to explore the underlying mathematical model. We conjectured that this would allow primary school students to reason more mathematically about rising speed. This interactive computer simulation of filling glassware has three components: a two-dimensional simulation of filling a glass with water from a tap, a number of predefined movable tick marks to turn the glass into a measuring cup (see Figure 2.1), and a graphing tool that supports both manual drawn and automatically generated graphs (see Figure 2.2). Next to the glass is a ruler to measure the height of the water; this ruler aligns with the vertical water height axis of the graph.

Making measuring cups—by dragging tick marks (labeled with the volume) to the corresponding places (heights) on the glass—allows students to reason about the co-varying quantities volume and water level height. They can express their understanding representationally with the graphing component. A student can choose between various drawing tools: plot a point, draw a straight line, or draw free-hand. Students may evaluate their measuring cups and graphs by running the simulation: as the glass is being filled the students can check if they placed the tick marks on the right spots on the glass. Similarly, the correct graph is drawn in red on top of the student's own graph (see Figure 2.2). The simulation offers the students instant feed-back and allows them to reflect on their work and underlying reasoning.

Each experiment consisted of three parts. In the introduction, the researcher informed the students about the experiment and asked a couple of questions about a measuring cup to start them thinking about filling glassware in terms of water level

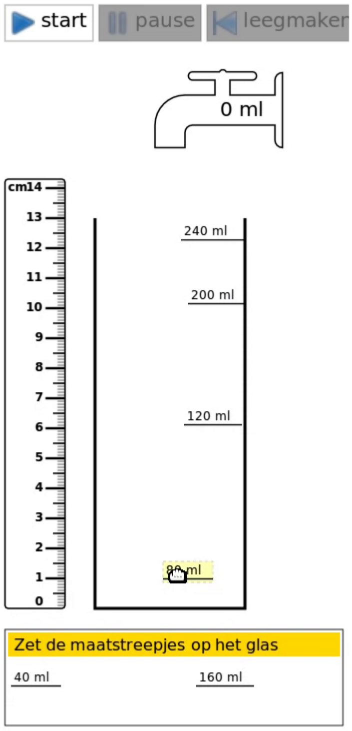


Figure 2.1: Creating a measuring cup of a highball glass: drag the given tick marks to the right places on the glass. (Experiment I)

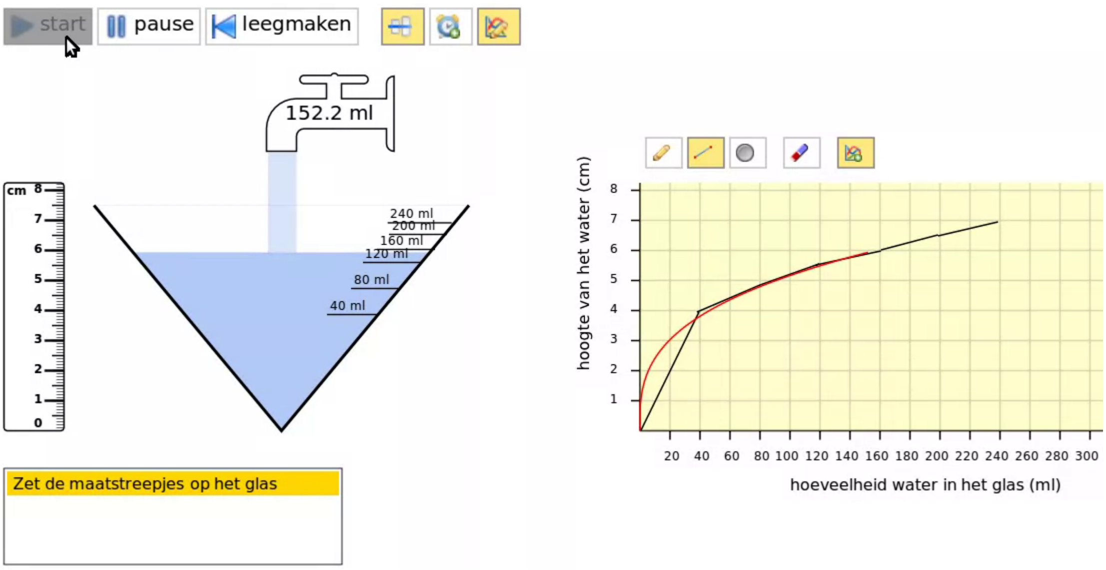


Figure 2.2: Evaluating drawing a graph of filling a cocktail glass: the red curve is the graph drawn by the simulation whereas the black line segments are user-drawn. (Experiment II)

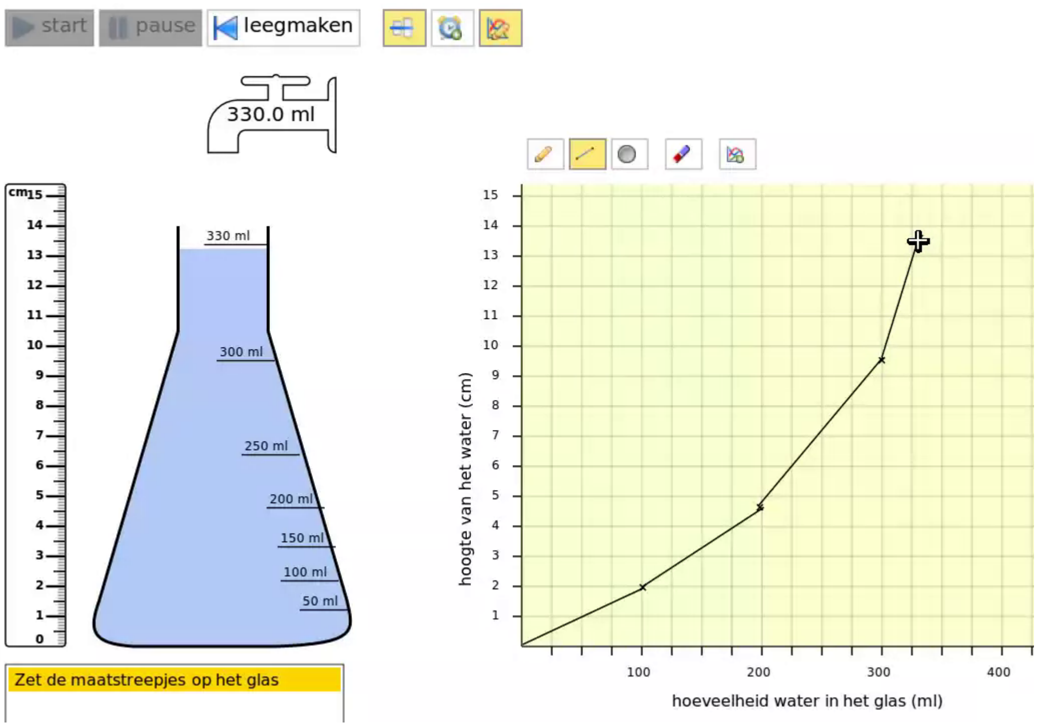


Figure 2.3: Drawing a graph of filling an Erlenmeyer flask during the pilot. The student did create a measuring cup first. (Experiment III)

height and volume. The second part consisted of three increasingly more difficult situations to explore: filling a cylindrical highball glass (see Figure 2.1), filling a cocktail glass (see Figure 2.2), and filling an Erlenmeyer flask (see Figure 2.3). Each problem was addressed the same way. The students were asked to create a measuring cup of the glass. Once they were finished, the researcher filled up the glass in the simulation and asked the students to evaluate their solution: was their measuring cup correct? Then the students were asked to draw a graph of filling the glass, followed by evaluating their solution after seeing the computer draw the correct graph on top of their own. While the students were performing these tasks, the teacher and researcher invited them to explain their reasoning. The experiments were concluded by an informal evaluation.

Data Analysis

The experiments were videotaped and the videos were transcribed. To capture the use of the software, the computer session was recorded using screen capture software. These screen capture videos were cut into smaller videos, one per task.

The analysis of the experiments was carried out in three steps. First, we analyzed the transcripts by categorizing all students' explicit verbal utterances about the co-variation of quantities volume and water height as indications of behavior on one of the five levels in the covariation framework. The students did not verbalize the basic understanding of level one—that there is change involving both volume and water height—, as understanding of this level was implicit in the context of the teaching experiment. At the second level, students would make statements similar to "the water is going up" indicating an increasing water level given increase in volume. At an understanding of level three, students would indicate that in some parts of the glass more or less water would fit.

Understanding at the fourth and fifth level would require the students to verbalize, respectively, awareness of rate of change while considering uniform increments of the input or awareness of instantaneous changes in the rate of change and how these would explain the shape of the continuous graph.

Table 2.1: Typical representational utterances during the tasks per level of reasoning in the covariation framework. Levels four and five would only be discernible in the measuring cup task when supported by appropriate verbal utterances.

	level	measuring cup task	graph task
L1	coordination	implicit	implicit
L2	direction	every next hash mark is put above the previous hash mark	a straight sloped line
L3	quantitative coordination	the distance between every two subsequent hash marks indicates the total amount of change of water height given increases in volume between these tick marks	the graph features certain points or line segments indicating the amount of change at these points or on these intervals
L4	average rate of change	as L3, discernible only when supported by appropriate verbal utterances	a compound graph of line segments

	level	measuring cup task	graph task
L5	instantaneous rate of change	as L3, discernible only when supported by appropriate verbal utterances	a continuous curve

Next, we analyzed the task videos by coding the representational utterances as behavior indicating reasoning on one of the five levels of covariational reasoning. Table 2.1 offers an overview of representational utterances per level and task. Representational utterances indicating reasoning on level one is implicit: the corresponding representational utterances are already contained in the set-up of the simulation. The task of creating a measuring cup has a maximum discernible level at level three for representational utterances unless higher levels of reasoning were indicated by corroborating verbal utterances about the rate of change between subsequent tick marks. The task videos were also coded for pseudo-behavior: when a representational utterance was not supported by verbal utterances, we coded them as pseudo-behavior.

Finally, the student's performance during a task as a whole was summarized as behavior on one level of reasoning.

Reliability and validity

To get an indication of the validity of the results, we used a peer debriefing strategy modeled after Barber and Walczak (2009). The goal of this peer debriefing was to reach a consensus with a second researcher external to our project about the analysis of the last problem of filling an Erlenmeyer flask in experiments 4 and 8. We agreed immediately about the results of the analysis of the 4th experiment. Reaching consensus about the analysis of the 8th experiment was more difficult, however. During the discussion, we studied also the analysis of experiment 1 and found that one of the utterances coded level four reasoning appeared to be an observation of what was happening in the simulation. It therefore should not have been coded. Getting back to the discussion about the analysis of the 8th experiment, we came to a consensus that reasoning at levels four and five was unlikely. Without additional indications of reasoning at these highest levels, the graphs categorized as indications of reasoning at levels four and five have to be classified as pseudo behavior.

Effectuation of the experiments

The experiments took place at a typical primary school located in a rural area in the south of the Netherlands. 8 one-on-one teaching experiments were carried out by the researcher (first author) and the students' teacher; nine[§] 5th graders (10 to 11 years of age; 4 boys and 5 girls) participated in the study. Because we were uncertain of students' reaction to the experiment, we chose to pilot the experiment with four above-average performing students, hoping that these students would react flexibly to last-minute changes. In the other five experiments, average to above-average performing students participated. All students are included in the analysis. The experiments took each between 25 and 45 minutes.

During the pilot, all students constructed a graph of filling the Erlenmeyer flask based on the measuring cup (see for an example, Figure 2.3). They drew lines from the coordinates of one tick mark in the graph to the next. The result was an approximation of the correct graph. However, it was unclear if this was an expression of the students' mathematical understanding of the situation or just an application of a known graph-drawing technique. To prevent the latter behavior in subsequent experiments, the measuring cup activity was removed from the third problem. Instead, the students were asked to describe the graph of filling the Erlenmeyer before actually sketching that graph. To further stimulate the students to sketch the graph as a whole, the axes were cleared of all ticks and labels except for the maximum values.

2.2.4 Results

We first discuss the results per problem and task, followed by a summary of the over-all results. In this discussion, the level of reasoning is put between parentheses.

Highball glass problem

Measuring cup task All students were able to construct correct measuring cups from the cylindrical highball glass (L3): all tick marks were placed approximately at the right height. Some students tackled the highball glass problem by repeatedly dividing the glass, while others first computed the distance between two tick marks before putting on all the tick marks separated by that distance.

Graph task Finishing the partially drawn graph of filling a highball glass was not a problem for any of the students (L3). Although some students just continued the

[§]In the third experiment, a pair of students participated instead of just one student.

line without regard for the situation (L2), e.g., without stopping at the maximum volume and water height. Other students drew line segments from the coordinates of one tick mark to the coordinates of the next, indicating quantitative coordination of water height and its corresponding volume (L3).

Cocktail glass problem

Measuring cup task In 6 experiments the students created a measuring cup from the cocktail glass with a linear scale similar to that of the highball glass (L2). They all initially expressed their surprise when they were confronted with the simulation of filling the cocktail glass, but soon they were able to explain why the cocktail glass filled up the way it did: because of the glass' shape there is less water at the bottom than at the top. During this evaluation phase their verbal statements were consistently level three (L3). In the other 2 experiments, the students created a measuring cup that took into account the non-linear characteristic of this situation (L3).

Graph task Drawing a graph of filling a cocktail glass was difficult for most students. In 5 experiments, the students drew graphs consisting of connected straight line segments, indicating that the water level rises less and less in each next segment. During the measuring cup task, in 4 of these experiments the students had created a (erroneous) linear solution (L2), but after seeing their cocktail glass being filled up, they realized that the graph would be different from that of the highball glass (L3). During the first experiment, the student, who created a non-linear measuring cup, indicated that the graph would start out slow and that the last segment of the graph would go steeper (L4). Although his graph does not have the correct concavity, he was the only student who talked about the relation between speed and steepness of the graph.

In the remaining 3 experiments, the students just drew a straight line (L2). They failed to coordinate the amount the water raised given the increase of the volume. In experiment 3, where a pair of students participated, the students did create a measuring cup that was rated at a higher level of reasoning (L3) than the graph. One of the pair realized that filling the cocktail glass was not a linear situation, but he was unable to explain his understanding to his partner. As a compromise, they drew a straight line graph (L2).

When these students were confronted with the curve drawn by the computer, they expressed their surprise for they expected the graph to consist of straight line segments. During the evaluation phase, the students' verbal statements were often of a higher level of reasoning than during the graphing task itself (L3), indicating their trouble representing their understanding graphically. For example, in the sixth experiment, the student connected the speed to the glass' shape and the steepness of the graph:

```
researcher: What do you notice?
student:    It is very fast here, and there it (unintelligible). Then the
            graph goes up as well. I think because it is narrower here and
            so it's filled up earlier. And then it is getting wider and it
            goes more slowly.
researcher: Yes. How do you see in the graph that it goes more slowly?
            How can you see that?
student:    Ehm, because the line is going up, the water, but here, below,
            somewhat faster, and there, at the top, somewhat slower.
```

Erlenmeyer problem

Graph task The students who did the revised Erlenmeyer problem (after the pilot), drew either a straight line (L2) or sketched some continuous curve (potentially L5). We expected the students to explain their curve by arguing something like: 'until its neck the Erlenmeyer's width becomes smaller and smaller all the time and therefore the water level rises faster all the time'. However, there was no such further verbal or holistic support to indicate that these continuous curves were an expression of understanding at level five, thus we coded this as pseudo-behavior (L5, pseudo).

In the 5^{th} experiment, for example, the student drew a continuous curve with the right concavity and a clear break in the shape of the graph to indicate the neck of the Erlenmeyer was also visible (L3)—even though the slope was incorrect. Given the limited graphing experience of the student this graph is quite a reasonable approximation of the correct graph. There is no additional indication however that this student drew a curve as an expression of understanding this situation in terms of instantaneous rate of change. Given that this student did see the correct graph of the cocktail glass just before—which was a curve—she might have used that knowledge in this situation. Most other students may have had similar reasons to try to draw a curve as a graph of filling an Erlenmeyer.

During the 8^{th} experiment, the teacher and researcher pressed the student to explain the reasons for drawing a curve in more detail. The student made a simple correspondence between a glass' shape and its graph: a straight shape results in a straight line as a graph and a non-straight shape in a non-straight line as a graph. Beyond that, the student could not explain why the graph is a curve. The teacher then continued with a thought experiment. What if the glass is malleable and its corresponding graph would consist of two connected straight lines of different slope, how would that glass look like? The student correctly figured that because there is a kink in the graph the glass itself should have an abrupt and sudden change in shape, like two stacked highball glasses of different diameter. The teacher concluded that if the glass has a conical shape, then the graph of filling that glass will be a curve. The student agreed, but could not explain why.

Even though students could indicate what part of a glass' shape would result in a curved line segment in the graph and what part would result in a straight line segment, they were unable to explain the curve phenomenon. This will partially be due to their lack of graphing experience and not having had any encounter with curves as graphs in school.

Results per experiment as a whole

Table 2.2: The students' level of reasoning in the covariation framework per problem. In the Erlenmeyer problem, most students exhibited pseudo-analytic behavior at a higher level. That pseudo-analytic level, if any, is put between parenthesis.

experiment	highball	cocktail	Erlenmeyer
I (pilot)	3	4	3 (4)
II (pilot)	3	3	3 (4, 5)
III (pilot)	2	3	3 (5)
IV	2	2	2 (-)
V	2	2	3 (5)
VI	2	3	3 (5)
VII	3	3	3 (5)
VIII	3	3	3 (5)

In Table 2.2 the students' levels of reasoning per problem (columns "highball", "cocktail", and "Erlenmeyer") are given. In three experiments (4 students) the level of reasoning seems to develop from (L2) to (L3) (experiments III, V, VI); in four other experiments (4 students) the level of reasoning stays constant at (L2) (experiment IV) or (L3) (experiments II, VII, VIII); while the level of reasoning of the student in the first experiment (an excellent student) on the cocktail glass problem exceeded that of his peers (L4). Pseudo-behavior at levels 4 and 5 during the graphing task of the Erlenmeyer problem is put between parentheses. Interestingly, even if classified as pseudo-behavior, most students showed verbal or graphical indications of reasoning at level 5, whereas there are no indications at level 4 except in the pilot, when the measuring cup was converted into a graph tick mark by tick mark.

Although the problems were increasingly more difficult, the participants' level of reasoning over the whole experiment remained quite consistent.

2.2.5 Findings

In answer to the research question, *How and at what level of the covariation framework come 5th graders to reason about two co-varying quantities in the context of filling glassware?*, we found that during the one-on-one teaching experiments the students came to reason at levels two and three of the covariation framework. The students were quite capable in estimating the relative amount of change at certain points or intervals. The students almost never talked about speed and when they did it had to be classified as pseudo-behavior. After the students had seen the non-linear situation of the cocktail glass and its continuous curve, they realized that the Erlenmeyer would also have a curved graph. However, the students were unable to characterize speed beyond observations of what they saw in the simulation. Once the students broke through the linearity illusion (de Bock, van Dooren, Janssens, & Verschaffel, 2002), most students came to reason at the quantitative coordination level (L3). Indications of reasoning at level 5, the instantaneous rate of change level, had to be classified as pseudo-behavior due to a lack of supporting evidence.

2.3 Conclusion and discussion

The one-on-one teaching experiments on filling glassware showed that the fifth grade students mainly reasoned at level 2 and 3 of the covariational framework (Carlson et al., 2002) that was designed for college students. A problem with using the covariation framework in primary school appears to be the limited vocabulary and graphing skills of the primary school students. Primary-school textbook tasks on speed typically concern calculational tasks on "average speed"—where average speed merely signifies the number of kilometers traversed in one hour for the students. This notion of speed fits well with the highball glass task, where the students spontaneously used the proportionality in the situation. The teaching experiment suggests that the connection with linear graphs was self-evident to the students, although we may wonder how deep this understanding might go, since this does not seem to be a topic that is touched upon much in primary school. The graphs in primary school textbooks are mainly limited to bar graphs and graphs consisting of line segments that connect data points. The required reasoning seems to be limited to comparing points in a Cartesian plane, or heights of individual bars in a bar graph. The question that arises is whether the covariation framework is not fit for investigating covariational reasoning of 5th grade students, or that the findings point to important weaknesses in the 5th grade students' capabilities.

Carlson et. al. define covariational reasoning in terms of mental actions (Carlson et al., 2002). These actions are intimately linked to behaviors (Carlson et al., 2002, table 1, page 357). While these behaviors are in turn linked to Cartesian graphs. Carlson et al. (2002) explain that the students' difficulties initially have been observed in the context of interpreting and representing graphical function information. However, we may argue that covariational reasoning is defined independently of the use of graphs, although the higher levels of the framework appear hard to reach without the support of graphs.

In our view, the students in our experiments who point out that the speed with which the water rises is determined by the width of the glass, and argue that the water rises slower when the (cocktail) glass gets wider, exhibit a form of covariational reasoning. They are unable, however, to express their covariational reasoning in a conventional graph. The main reason being that the students lack basic graphing capabilities. Moreover, they also lack a sound understanding of speed; for speed (whether in the context of rising water or moving objects) is not yet a variable for them. They can reason about speed qualitatively, using adjectives such as "fast" and "slow", but they are not yet familiar with speed as a magnitude.

We do agree with Carlson et. al. that what we should aim for is covariational reasoning in the context of interpreting and representing graphical function information. In this respect, the 5th grade students' lack of graphing capabilities is not so much a matter of limitation of the covariation framework, but an issue that has to be addressed if we want to teach those students about covariational reasoning. The graph of a function offers a powerful image of a function as a set of ordered number pairs (Sfard, 1991). It affords thinking dynamically, imagining a point moving along the graph. By extension, one may think of a tangent line moving along the graph of a function, which can be done with a computer simulation.

2.3.1 Implications for the design of the instructional sequence

The teaching experiments offered us more than just the findings about the students' level of covariational reasoning. Through designing, performing, and analyzing the teaching experiments, we did gain additional information about the potential starting points of the instructional sequence on learning to reason about instantaneous speed in the context of filling glassware. While working on the task of creating a measuring cup from the highball glass, the students showed that they were very familiar with linear proportionality. They also linked linear proportionality to a linear graph, suggesting an implicit notion of constant speed. We have to caution, however, that students tend to apply the linearity prototype everywhere (Van Dooren, Ebersbach, & Verschaffel, 2010). On the other hand, the students showed to easily break through this "linearity illusion" (de Bock et al., 2002) when confronted with the simulation of filling the cocktail glass. Once that had happened, the students showed to understand the relationship between a glass' width and rising speed. We further learned that students this age know graphs mainly as consisting of points connected with straight line segments. Seeing the computer draw a continuous graph while filling a cocktail glass made them realize that continuous graphs better describe continuous change, but they could not express this relation verbally.

We concluded that we might try to capitalize on the students' understanding of the relation between a glass' shape and the rising speed—the wider a glass, the slower the water rises. Given this insight, students might also realize that the instantaneous speed in a cocktail glass at a given height is equal to the instantaneous speed in an highball glass that has the same width as the cocktail glass at that height. The highball glass can then become an indicator of the instantaneous speed of rising at any given point in the cocktail glass or any other glass. Basically, this would be the same type of definition William Heytesbury gave for instantaneous velocity around 1335: the distance that would be traveled if the speed would stay constant for a given period of time (Clagett, 1959).

This notion of instantaneous velocity can be translated to the context of filling glassware as follows (see Figure 2.4 on p. 36): By equating the *instantaneous* rising speed in the cocktail glass with the *constant* rising speed in an (imaginary) highball glass. Linking the speed in an arbitrary glass with the speed in a highball glass would eventually enable students to quantify the instantaneous speed in a point by computing the corresponding highball glass' constant speed. Building on the correspondence between a highball glass's graph and the steepness in a point of a curve, the students may construe the tangent line in a point of a curve as an indicator for the instantaneous speed in that point. Given students' familiarity with constant speed and graphing linear situations, students then may quantify the instantaneous speed by computing the rise over run of the tangent line.

We used this information to set up a series of design experiments, which we discuss elsewhere (Chapters 3–5).

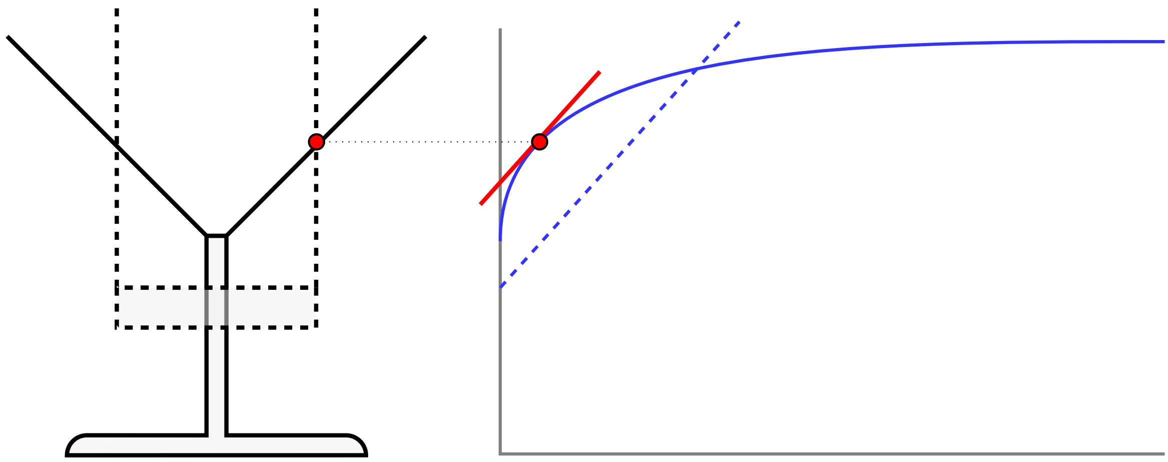


Figure 2.4: The instantaneous rising speed in the cocktail glass at the red point is the same as the constant rising speed in the highball glass; the highball's graph, the straight dashed line, is the tangent line on the cocktail glass' curve at the red point. (This image is taken from de Beer, Gravemeijer, and van Eijck (2015))

3

Discrete and continuous reasoning about change in primary school classrooms[*]

3.1 Introduction

In answer to the call for new elementary science and technology education for the 21st century (Léna, 2006) we started a design research (DR) project to get a better understanding of how to teach calculus-like topics in primary school. We think teaching this topic is important because the interpretation, representation, and manipulation of dynamic phenomena are becoming key activities in our high-tech society. We want to start working on this in primary school, by supporting students in developing a sound mathematical understanding of rate of change. Research suggests that with support of interactive simulations of dynamic phenomena, younger students are able

[*]This chapter is published as de Beer, Huub, Koeno Gravemeijer, & Michiel van Eijck (2015) 'Discrete and continuous reasoning about change in primary school classrooms' *ZDM Mathematics Education*, doi:10.1007/s11858-015-0684-5

to reason about rate of change without relying on mathematical concepts that are normally taught in secondary education (Thompson, 1994b; Stroup, 2002; Kaput & Schorr, 2007).

Trying to understand how to teach a topic in primary school that, conventionally, is taught in high school or college fits DR (Kelly, 2013). We follow the DR approach outlined in Gravemeijer and Cobb (2013): DR involves conjecturing an initial local instruction theory (LIT), and elaborating, adapting, and refining this LIT in multiple design experiments (DE). Although this type of DR has been characterized as "validation studies" (Plomp, 2013), the aim is not merely to treat the LIT as a hypothesis to test in a DE, but also to explore innovative learning ecologies and students' learning processes (Gravemeijer & Cobb, 2013).

Typical for DR is the aim to also generate new ideas. In relation to this, we may refer to Smaling's (1992, 1990) conception of objectivity as a methodological norm:

> (...) the researcher must strive to do justice to the object under study; (...) doing justice has two important aspects: the positive aspect, which concerns the opportunity for the object to reveal itself, and the negative aspect, concerning the avoidance of a distortion of the image of the studied object. [Smaling (1990), p. 7; translation by the authors]

Avoiding distortion may be associated with reliability in quantitative research. Smaling (1990) observes that reliability refers to the absence of accidental errors and is often defined as replicability. He argues that for DR this can be translated into virtual replicability, or "trackability"—which requires that the research is reported in such a manner that it can be reconstructed by other researchers. In DR, Smaling's positive aspect requires one to look for signs that may indicate possibilities for promising alternatives. In a more general sense, letting the object reveal itself, or "letting the object speak", relates to the process of generating new conjectures to explain what happened during one or more DEs. Generating new explanatory conjectures is a process known as "abductive reasoning," which is to "rationalize certain surprising facts by the adoption of an explanatory hypothesis" (Fann, 1970, p. 43), and is employed in the retrospective analysis of a DE. Surprising facts may indicate that our understanding of what happened misaligns with the students' learning process during the DE. To resolve this distortion, new conjectures are formulated. Subsequently, these new conjectures are tested against the data collected during the DE. Although "testing conjectures" might call up an association with testing hypotheses in experimental research, it means something different here. In line with Glaser and Strauss' (1967) method of constant comparison, it refers to the re-examination of the data to establish the extent to which these conjectures actually describe the data set.

In this chapter, we illuminate this abductive process on two different levels. We describe how, during the retrospective analysis of the third DE, we refined the LIT by generating the new explanatory conjecture that primary school students come to the classroom with a continuous conception of speed and only switch to discrete reasoning because of a lack of means for visualizing continuous change. This, in turn, led us to realize that in our project average rate of change is a hindrance rather than a necessity in teaching instantaneous rate of change in primary school. We start by discussing the theoretical background of the 3rd DE and placing it in the context of our DR project. We continue by describing the unexpected event that triggered the abductive process, which is followed by the description of the retrospective analysis and its results. We conclude with a discussion of the findings and the place of generating theory in DR.

3.2 Theoretical background

3.2.1 Qualitative calculus

We chose to expand on the "qualitative calculus" approach (Stroup, 2002) to teaching calculus-like concepts in primary school. Stroup (2002) argued that the qualitative calculus is a worthwhile enterprise in itself, and not merely a preparation to conventional calculus courses. Qualitative calculus differs in two important respects from conventional approaches to calculus: by (1) starting with non-linear situations of change instead of linear situations and (2) by developing a non-ratio-based understanding of rate of change (Stroup, 2002). We will elucidate both points.

The linear prototype seems to be over-used in school situations, resulting in students seeing and applying linearity everywhere (de Bock et al., 2002; Ebersbach, Van Dooren, & Verschaffel, 2011), even though linear progressions are uncommon in real situations. Moreover, linear situations may be too simple to support students to develop their understanding of calculus-related topics (Stroup, 2002). It may therefore be better to start with non-linear situations, especially since primary school students do have an intuitive understanding of non-linear situations (Ebersbach & Wilkening, 2007) and are able to reason about these situations (van Galen & Gravemeijer, 2010; Stroup, 2002).

Non-ratio-based understandings of rate are usually interpreted as an earlier form of understanding, which precedes that of ratio-based understanding (Stroup, 2002). The formal definition suggests that understanding instantaneous rate of change has to build on understanding average rate of change, which in turn is based on a ratio. It further suggests that students have to know about functions, algebra, and limit to be able to fully understand instantaneous rate of change. However, Stroup offers an alternative approach to learning rate, in which students construct

> an intensive understanding of rate ("fastness" independent of a particular amount of change or amount of time) that is both *powerful*, yet *not organized as a ratio of changes*. (p. 170)

In our research we want to expand on this qualitative understanding and support primary school students in also developing a quantitative understanding of instantaneous rate of change that does not involve the difficult limit concept (Tall, 1993). We want students to develop a holistic understanding of change as a continuous process and develop an understanding of rate of change (or speed) at a point in time. We assume that using and interpreting Cartesian graphs may support students in developing such a dual understanding, because analyzing the shape of a curve may give rise to reasoning about continuously changing speed qualitatively, while reasoning about speed at a point or over an interval could give rise to quantitative reasoning about rate of change.

3.2.2 Graphing

However, graphing is a marginal topic in primary school (Leinhardt et al., 1990) and, even for older students, graphing appears far from trivial (Leinhardt et al., 1990). Mevarech and Kramarsky (1997) reported that they developed a graphing course that, although it did reduce middle-school students' superficial graphing errors, it did not affect their more fundamental alternative conceptions about graphing. To overcome these problems, Roth and McGinn (1997) propose a social-cultural perspective focusing on students becoming graphing practitioners. This resonates with diSessa et al. (1991)'s study of students' meta-representational competency (diSessa & Sherin, 2000) in which students did invent all sorts of idiosyncratic representations of motion. Building on this research, Nemirovsky and Tierney (2001) suggest an approach in which students' invented representations of motion are linked to conventional representations. This raises the question to what extent students should be told how to graph or invent graphing for themselves. For one thing, students will have seen and used graphs before entering any formal graphing education (Mevarech & Kramarsky, 1997). Furthermore, some argue that by using "transparent tools" (Hancock, 1995) such as computer generated graphs, students are able to start exploring dynamic phenomena without needing to know graphing conventions. However, Meira (1998) emphasizes that there is no inherent transparency to instructional representations, and that these representations are "meaningful only with respect to learners' activities" (Meira, 1998, p. 140). Nevertheless, Ainley et al. (2000) claim that students can learn graphing conventions by "active graphing", that is, by using computer generated graphs as exploratory devices to drive experimentation and data analysis (Pratt, 1995; Ainley et al., 2000).

In summary, we may conclude that it is advised to aim for an active role of the students in interpreting and, preferably, inventing graphs, while computer simulations and computer generated graphs may play a supporting role.

3.2.3 Emergent modeling

We place our research in the tradition of the domain-specific instruction theory of realistic mathematics education (RME). Two of its design heuristics, guided reinvention and emergent modeling (Gravemeijer & Doorman, 1999; Gravemeijer, 1999), inform the development of our LIT. According to the first heuristic, students are to be supported in reinventing mathematics by solving specifically designed tasks under guidance of teachers (Gravemeijer & Doorman, 1999). In line with the second heuristic, emergent modeling (Gravemeijer, 2007), models are designed to support students in reinventing more formal mathematics. The model originates from students' informal mathematical activity. Working with a *model of* their informal mathematical activity, students subsequently develop the mathematical relations that enable them to conceptualize the model differently with the help of the teacher. Instead of signifying their informal activity, the model starts to signify the mathematical relations. In this manner, the model becomes a *model for* more formal mathematical reasoning (Gravemeijer, 1999).

3.2.4 Continuous and discrete

Whereas the above offers footholds for the design of the course, literature on the dichotomy between discrete and continuous perspectives may offer footholds for analyzing interaction processes in the classroom. Castillo-Garsow (2012) introduced a three-fold classification whereby students perceive a problem and method as either continuous or discrete, come up with a solution that is either continuous or discrete, and can use their discrete or continuous reasoning to use that method to come to the solution. They introduced the concepts "chunky" and "smooth" to characterize the two forms of reasoning (Castillo-Garsow, 2012; Castillo-Garsow, Johnson, & Moore, n.d.). Thinking about change in terms of intervals, or completed chunks, is called "chunky." Students with a chunky image of change see change on an interval as the end-result of change on that interval. Students that see change as a continuous process, however, have a "smooth" image of change (Castillo-Garsow, 2012; Castillo-Garsow et al., n.d.). Smooth thinking is fundamentally different from chunky thinking, and chunky thinkers will have trouble thinking about change as a continuous process. Thinking in terms of chunks from a smooth perspective, however, seems more easily achievable (Castillo-Garsow et al., n.d.).

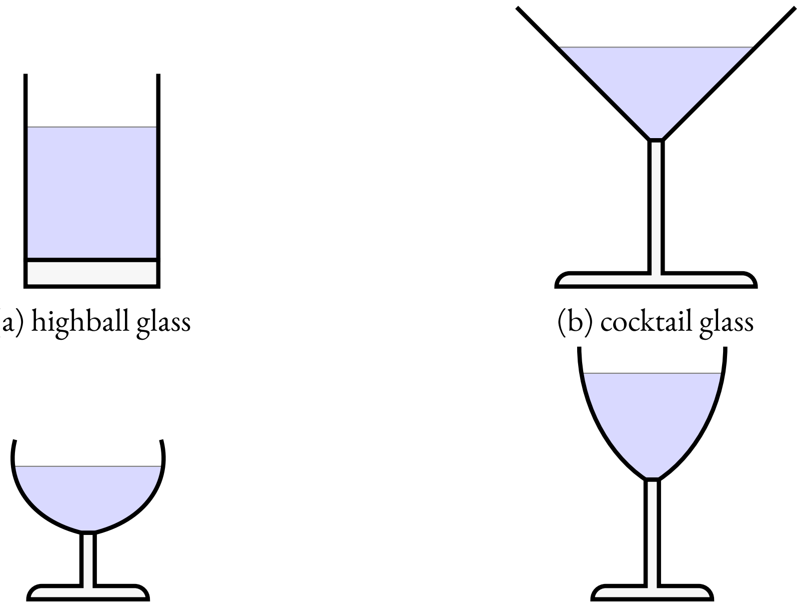
Although students experience continuous change from an early age on, discrete reasoning seems to take prevalence in mathematics education (Castillo-Garsow, 2012). At the same time, students seem to have trouble representing continuous motion and often opt for discrete graphs (McDermott, Rosenquist, & Zee, 1987). Nevertheless, successes have been reported of young students moving from discrete representations of motion to continuous representations (diSessa, 1991). Other research suggests that, when carefully embedded in an instructional design, primary school students can move between conflicting visualizations of the same phenomenon and, as a result, create a deeper understanding of that phenomenon (Abrahamson, Lee, Negrete, & Gutiérrez, 2014). However, interpreting continuous representations of change, such as curves, is far from trivial (Saldanha & Thompson, 1998); through interval analysis students can interpret a curve discretely (Nemirovsky, 1994).

3.3 Design experiment 3

Before discussing DE3 in more detail, we place it in the context of the overall DR project by discussing the first two DEs. After that, we present the LIT succinctly, followed by an overview of the instructional sequence and describing the educational environment in which we tested that instructional sequence. We conclude with detailing the first lesson and a half.

3.3.1 Prelude

Early in our project, we chose to aim at developing a LIT that delineates a possible learning process for primary school students to develop their understanding of instantaneous rate of change in the context of filling glassware (such as depicted in Figure 3.1). We chose this context because of students' familiarity with filling glasses, and the room for exploring multiple non-linear situations, which fits the qualitative calculus approach. The origin of this context can be traced back to Swan (1985), who used it in a secondary mathematics text-book on functions and graphs. Since



(a) highball glass (b) cocktail glass

(c) cognac glass (d) wine glass

Figure 3.1: Various glassware

then it has been used by others in middle school (McCoy et al., 2012) and secondary education (Castillo-Garsow et al., n.d.; Johnson, 2012). We add to this literature by using this type of problem to foster primary school students' understanding of instantaneous rate of change. In the remainder of this chapter, unless indicated otherwise, we abbreviate "speed of rising water" by the term "speed."

To prepare for DE1, we explored the problem domain by performing eight one-on-one teaching experiments with above average performing 5th graders. Based on what we learned, we were able to formulate an initial LIT for DE1. After developing this LIT into an instructional sequence, we tried it out in a gifted 5th/6th grade classroom with an experienced teacher. During the retrospective analysis, we identified two major problems with our approach to teaching instantaneous speed: a learning environment that promoted chunky thinking and a lack of conceptual discussion of instantaneous speed.

To overcome these problems, we reorganized the LIT in two ways DE2. First, to encourage smooth thinking, we removed activities that had promoted discrete thinking during DE1. Furthermore, instead of starting the exploration of filling glassware with the highball glass, we hoped that starting with the non-linear situation of the cocktail glass would encourage students to see the changing speed. Second, to improve conceptual discussion of speed, we opted for a modeling-based learning (MBL) approach for students to both rediscover the curve as a fitting model to describe non-linear situations as well as to explore the situation of filling the cocktail glass more deeply.

To test these conjectures, we performed a small scale teaching experiment with four 5th graders. Although we did get more clarity as to what extent students' representational competency reflected their understanding of filling glassware, the students created increasingly more discrete models during the modeling activities. They were unable to switch to continuous models without our guidance. However, once the curve was introduced, they were able to connect it to their understanding of the situation and, with reference to the graph, they could describe the continuous changing speed. That understanding transferred to their subsequent exploration of the linear situation of the highball glass as well.

3.3.2 LIT

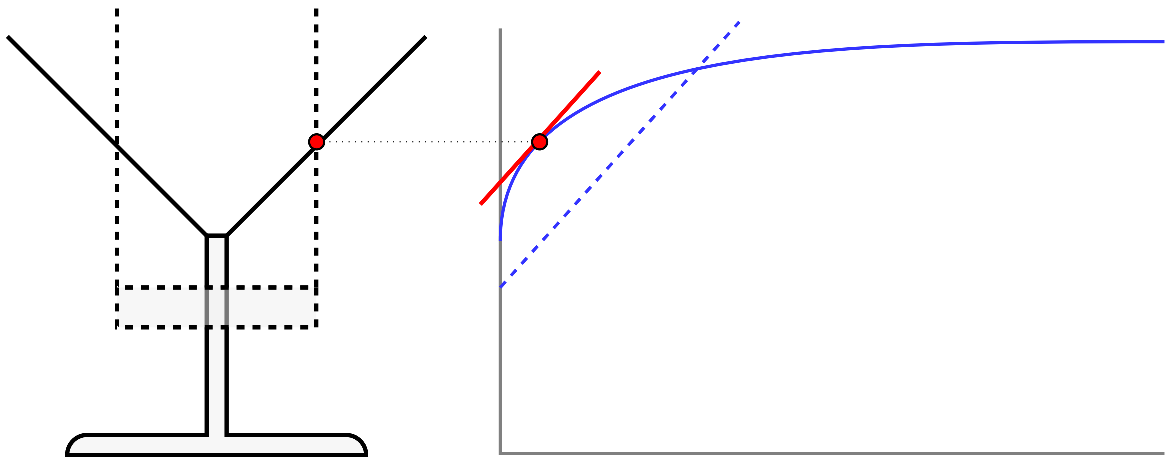


Figure 3.2: The *instantaneous* speed in the cocktail glass at the red point is the same as the *constant* speed in the highball glass (dashed line); the highball's graph, the straight dashed line, is the tangent line on the cocktail glass's curve at the red point

We took what we learned during DE2 as a basis for refining a learning trajectory that forms the core of the LIT as follows. Given a cocktail glass, students are given the task to draw how the water height changes when the glass fills up. After observing it fill up, they notice that the water level rises slower and slower, and they realize this is the result of the glass's increasing width. This realization allows the students to form valid expectations about the process of filling glassware and they come to depict it both as a discrete bar chart as well as a continuous graph.

We expect the students to link the curve of the continuous graph with the continuous change of the speed of the rising water: at every moment that speed is different.

From this perspective, students come to interpret the speed of rising as an instantaneous speed. The learning trajectory focuses on deepening that concept of instantaneous speed both qualitatively as well as quantitatively by exploring two avenues of thought. First, by comparing the speed in the cocktail glass with the constant speed in a cylindrical highball glass to answer the question of when the water rises with the same speed in both glasses (see Figure 3.2, left hand side). The constant speed of an imaginary highball glass becomes a measure for the instantaneous speed in the cocktail glass. Second, building on that understanding, trying to measure speed in a graph by interpreting the straight line graph of the highball glass as a tangent line on the curve of the cocktail glass (Figure 3.2, right hand side). Throughout this process, the representations of the speed in the highball glass act as an emergent model of measuring instantaneous speed.

3.3.3 Instructional sequence, educational environment, and data collection

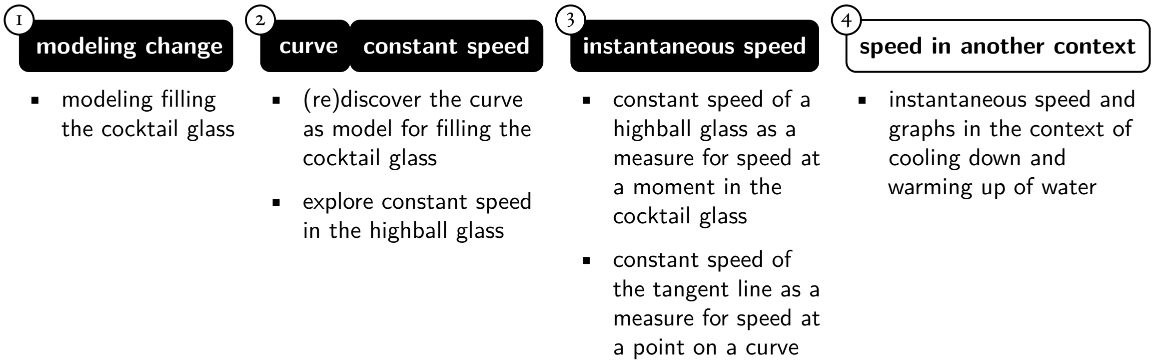


Figure 3.3: Overview of the actual lessons in DE3

Based on this LIT, we developed an instructional sequence of four lessons (see Figure 3.3). The first three lessons were tested in three subsequent weeks. In the first lesson and a half, we intended the students to rediscover the curve as a fitting model of filling the cocktail glass. The second half of the second lesson focused on students exploring and quantifying constant speed when filling various highball glasses. Then, in the third lesson, building on students' understanding of the curve and constant speed, we intended for students to construct the twofold understanding of instantaneous speed as delineated in the LIT. The fourth lesson took place after a break of a month and was about rate of change in a setting of heating and cooling.

We tried this instructional sequence at a school for gifted children in the South of the Netherlands. This school was located at the headquarters of a municipal school collective and only offered the gifted program to selected students from schools of the collective. Twice a week, these students would come from all over the municipality to attend the program for half a day during normal school hours. The gifted program focused on students' creativity and social-emotional development whilst offering an intellectually challenging environment with topics in the area of language and culture. There were two groups of 24 gifted students from grades 4-6. Classroom 1 (C1) participated in our study on four Friday afternoons (7 girls, 17 boys; average age: 9.75 years; average grade: 5.17) and classroom 2 (C2) participated on four Monday afternoons (6 girls, 18 boys; average age: 9.5 years; average grade: 5). Each lesson was tried in C1 first and then, after the weekend, in C2.

Normally, both groups were taught by the same pair of teachers. However, only one teacher participated in our study. He was a novice teacher with less than three years of experience who also worked as a project manager and teacher of "media literacy" at the teacher training institute of a nearby college. He confessed to an affinity for science and technology; in high school he followed the Science and Technology track which included basic calculus. The teacher's role was essentially executive in nature. We trusted in his expert opinion to follow the planned lessons, while fitting them to his two classrooms, and to initiate and support suitable classroom social norms for MBL. He created and supported a classroom culture where students felt free to express their opinions, ask questions, and indicate their doubts or disagreements.

During the test we gathered video recordings, screen capture videos, student products, observations, and audio recordings of meetings with the teacher about the lessons. The video-taped whole-class discussions were transcribed.

3.3.4 Testing the instructional sequence: rediscovering the curve

In this chapter, we report only on the first lesson and a half wherein we intended the students to construe the curve as a fitting model to describe filling the cocktail glass. In the first lesson, the students modeled filling a cocktail glass four times (see Figure 3.4): 1) each student depicted how he or she imagined the glass to fill up; 2) students paired up to create a new model together; 3) after observing an actual cocktail glass fill up in front of the classroom, the students improved their models in groups of four; 4) in pairs the students explored filling the cocktail glass in detail using a computer simulation. After each activity, students' models were discussed in class.

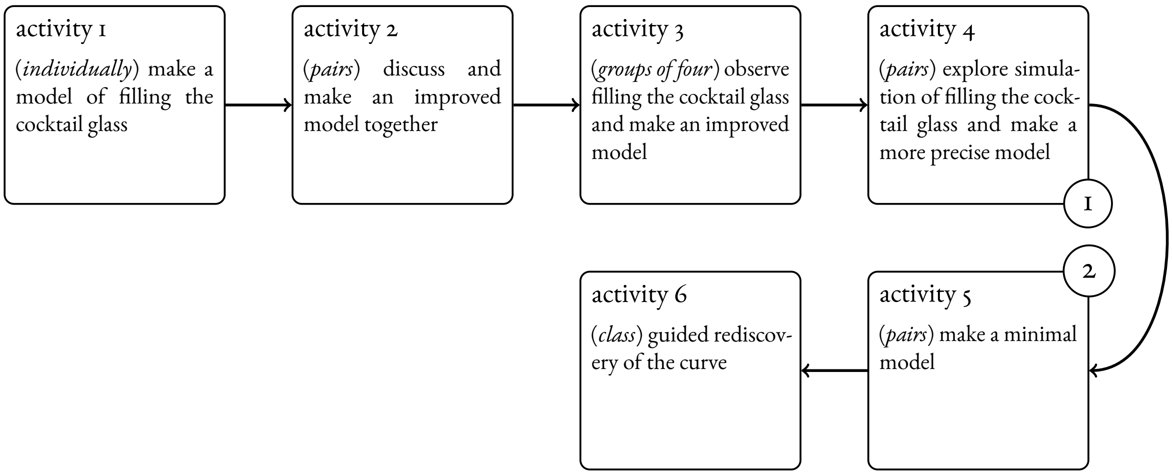


Figure 3.4: Rediscovering the curve as a fitting model of filling the cocktail glass with six successive modeling activities spread over two lessons

In the second lesson, the students were asked to create a minimalist model (activity 5). Students tried to remove everything that they deemed not absolutely necessary to represent filling the cocktail glass. In both classrooms there were one (C2) or two (C1) models that included continuous characteristics: the students connected points or intervals together into something resembling a line graph (Figure 3.11). Next, the teacher invited multiple pairs to present their model. In C1, the line graph was among the presented models; in C2, it was not.

In both classrooms, the teacher moved the discussion towards a more mathematical conception of speed by building on one of the models presented (activity 6). Beforehand, we had conjectured the following learning trajectory from students' minimal models towards students discovering the curve as a fitting model for filling the cocktail glass (see Figure 3.7):

Under the guidance of the teacher, the minimal models are condensed into graph-like representations first and Cartesian graphs later. Vertical value bars (adapted from Bakker and Gravemeijer (2004); Figure 3.8) come to signify water heights in the glass at specific moments in time, and arrows connecting these points come to signify change (see Figure 3.8). In this phase, the graph—as a model—still derives its meaning, for the students, from its reference to the actual situation of the cocktail glass that is being filled (either in reality or in a computer simulation). Reflecting on continuous graphs of computer simulations and student-generated graphs, and by discussing the relation between the shape of the glass, the rising speed of the water, and the shape of the graph, students are expected to come to see the curve as signifying both the

Figure 3.5: Margaret's realistic depiction of filling the cocktail glass. She annotated it with "It is filling with wisky" (C1)

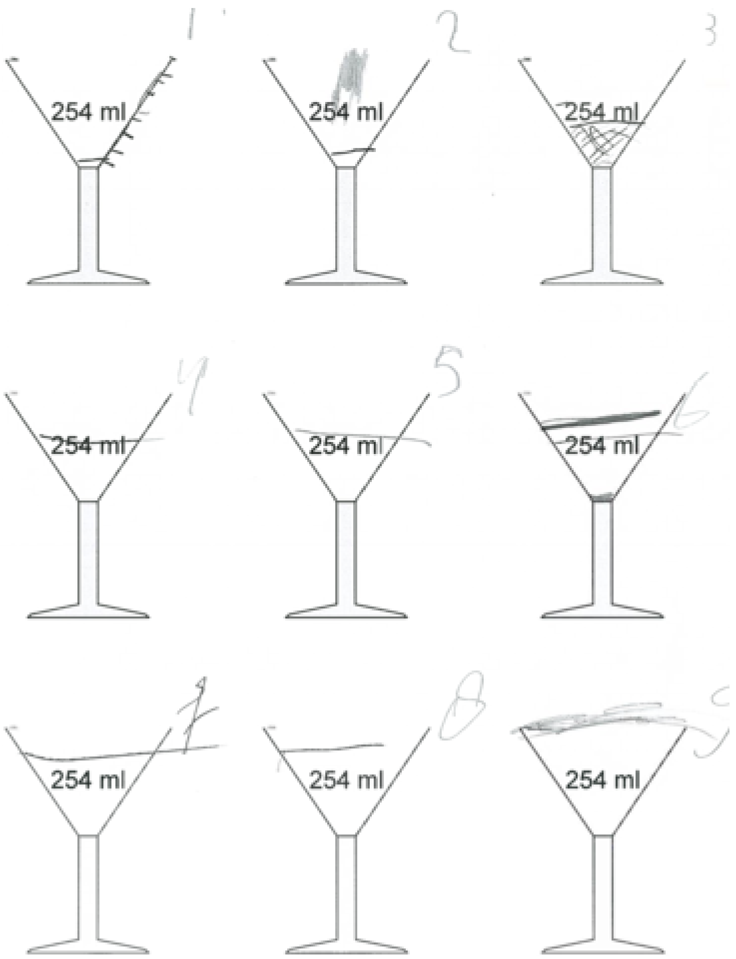


Figure 3.6: Kevin, Eric, Ryan, and Andrew's snapshots model (C1)

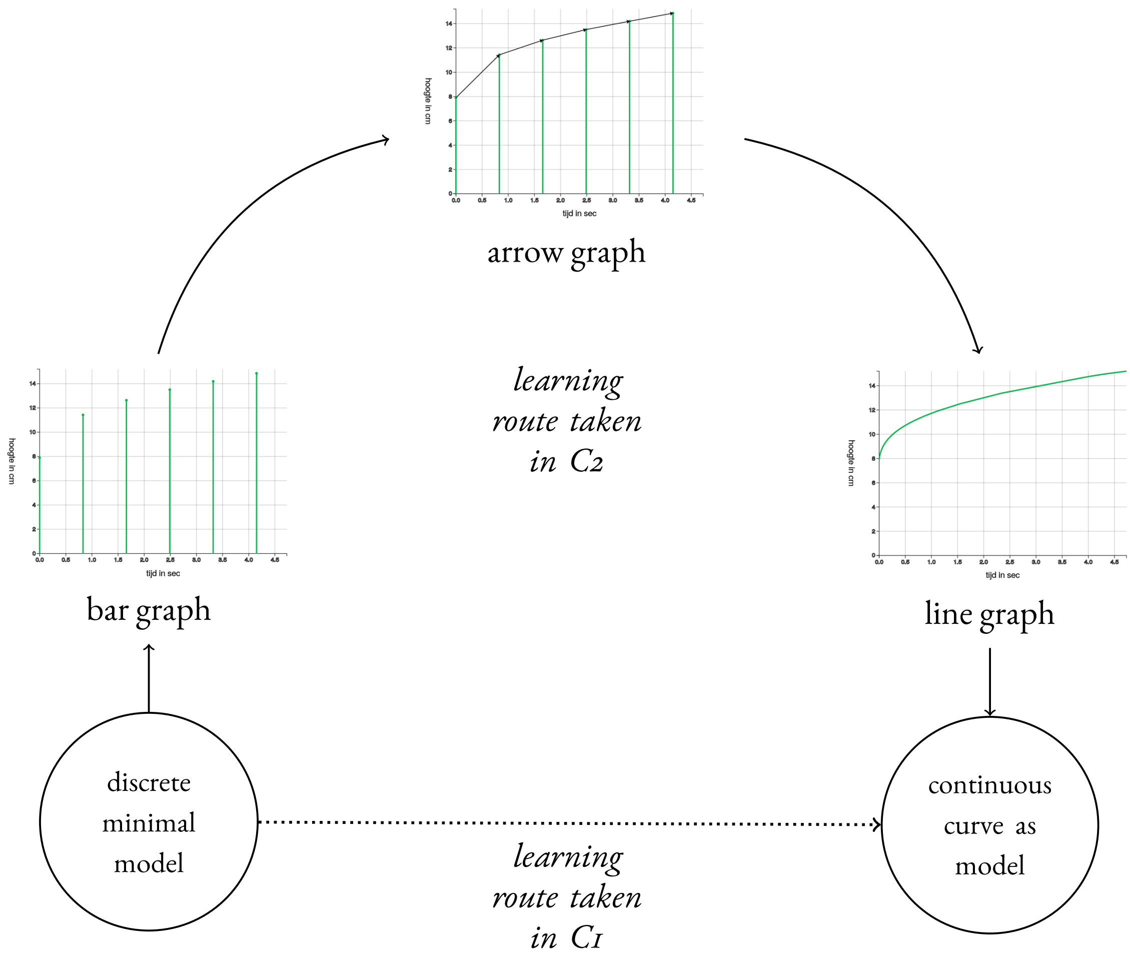


Figure 3.7: Conjectured learning trajectory. The teacher guides the class to rediscover the curve as a model for filling the cocktail glass in three steps: 1) connecting students' minimal models to a bar graph; 2) indicating the change between bars through the arrow graph; 3) recognizing the continuous nature of change in the line graph. In C2, the conjectured route was taken (solid arrows), whereas in C1 the students discovered the curve before taking that conjectured route (dashed arrow)

changing value and the rate of change. The model then has become a model for more formal mathematical reasoning.

Although students adopted the curve in both classrooms, the actual learning trajectory of activity 6 evolved quite differently in the two classrooms. In C2, the learning trajectory followed our conjectured trajectory. The students' reasoning was mainly discrete and proved hard to redirect towards continuous reasoning. In C1, however, the reasoning was immediately continuous, resulting in an unexpected actual learning trajectory. We therefore decided to focus the retrospective analysis on discrete and continuous thinking.

3.4 *Retrospective analysis*

For the retrospective analysis, we used the two-step method described in Gravemeijer and Cobb (2013) which builds on grounded theory (Glaser & Strauss, 1967), in particular the elaboration of Cobb and Whitenack (1996) on this method was used. The first phase focuses on *What happened?*. During this step conjectures about what happened are formulated and tested against the whole data set. In the second phase, based on the results of the first phase, conjectures about *Why did this happen?* are formulated and tested against the data. This retrospective analysis provoked us to re-evaluate our prior assumptions about students' discrete reasoning and to generate a new idea on how to teach instantaneous speed in primary school.

3.4.1 Phase 1: What happened?

We formulated seven conjectures about what happened, which we tested against the data.

Conjecture 1 Even though some students use snapshots to describe the filling process in the first series of activities, their annotations and utterances indicate that they use those snapshots to describe the process of change, not individual data points.

During the first two activities, most students created a realistic depiction of the situation (like Figure 3.5) or made a more abstract model focusing on one aspect or another. Many students annotated their models with elements such as air bubbles, waves, or droplets, and during the discussion they explained that they added these elements to enhance the realism of their models. For example, Susan explained:[†]

[†] '(?)' means incomprehensible, '(...)' means a cut in the transcripts made by the authors, and other remarks between

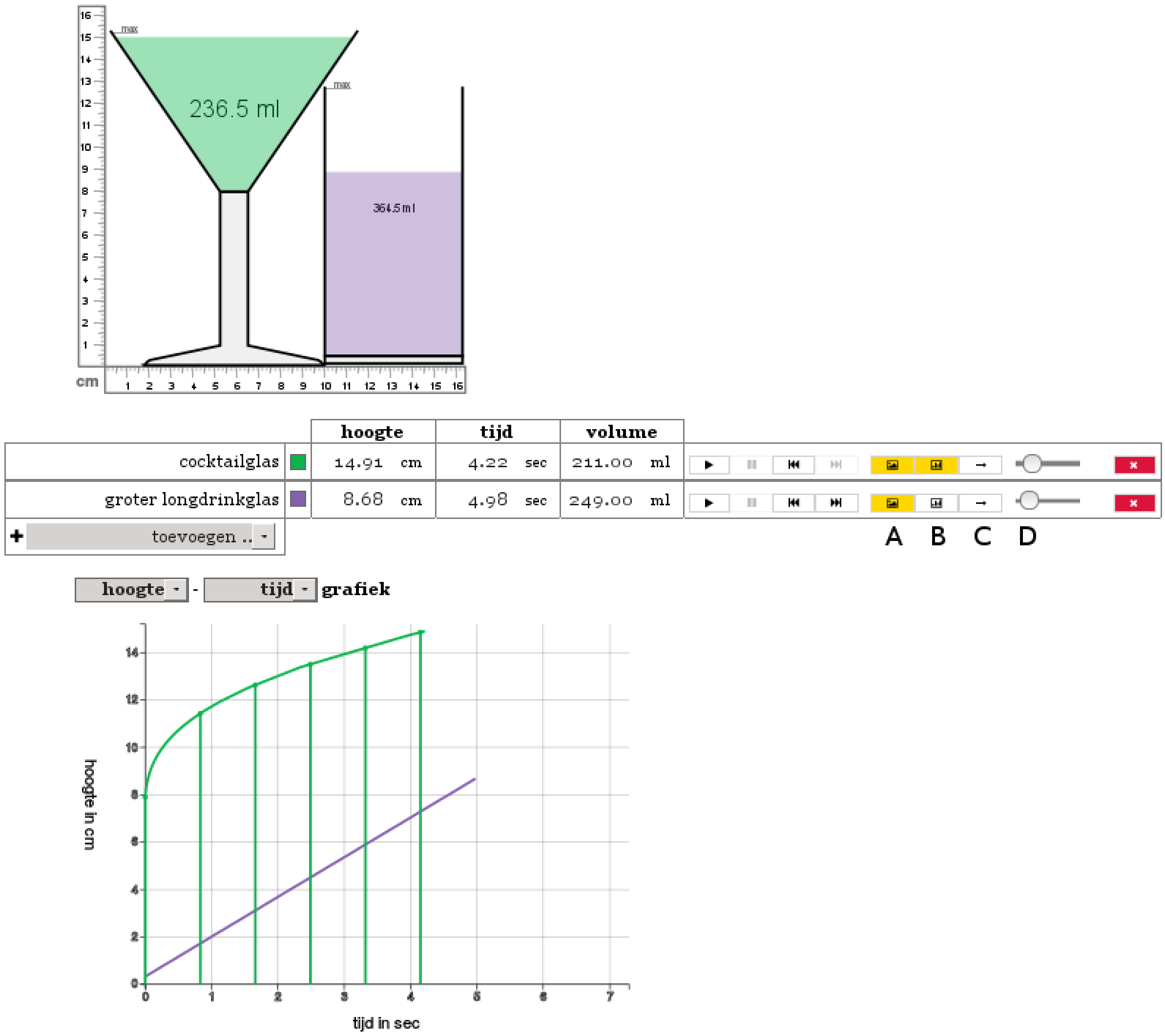


Figure 3.8: Interactive simulation software with graphing functionality showing graphs of the cocktail glass (green) and the highball glass (purple). Three different kinds of graph can be shown or hidden through buttons A, B, and C, respectively the bar graph, the arrow graph, and the line graph. With slider D the step size or interval in between the bars and the arrows can be changed, showing more or fewer points

```
Susan   : (...) Here it is below the water line and air bubbles appeared
          everywhere, like, for example, if you pour in water, you often
          see these air bubbles appear
```

Eight of 43 models made in activity 1 represented change of a quantity over time by using a sequence of snapshots. Eric explained his choice for using multiple snapshots:

```
Eric    : Because otherwise (?), it seems as if it is immediately, you turn
          on the tap, and immediately everything is all filled up. That is
          not so clear
```

To him, modeling the situation as one big glass did not tell the whole story, but multiple snapshots did. Later, another student agreed with him, arguing that his snapshots model was "more real," but he was unable to explain why.

One student explicitly modeled speed in her snapshots model (Figure 3.9) by labeling the snapshots from "fast" to "slow." Another student annotated the subsequent snapshots with increasingly bigger arrows to denote a growing pressure on the sides of the glass while it filled up. These students seemed to use the snapshots model to denote the *process* of change, not to denote *specific* data points that somehow were of interest to them.

While discussing students' second model, the teacher asked what they could do with a snapshots model. In C2, students also understood the snapshots to denote a process. To Richard, the model showed that:

```
Richard : That when you pour in more, it gets wider
Teacher : (...) But what can you say about the speed of it, for example?
          What can you say about the rising? Paul?
Paul    : Because, eh, it goes very fast in the beginning and the higher
          it gets, the slower it goes
```

Paul realized that the snapshots model could be used to show the changing speed. Most other students came to the same realization only after seeing an actual cocktail glass fill up (conjecture 2). However, after that, the students started to strongly prefer the snapshots model. In activity 3, 13 of 15 models were discrete models and in activity 4 all models were either a snapshots model or a table.

Conjecture 2   When the students see an actual cocktail glass fill up, they realize that the constantly decreasing speed is caused by the increasing width of the cocktail glass.

When the students saw the cocktail glass fill up, their comments showed that they realized that it is a non-linear situation and that the speed decreased due to the increase of the glass's width. For example, in C1, two students explained:

parentheses are observations of what happened on the video at that point.

Figure 3.9: Linda's discrete snapshots model co-varying time, volume or water level, and speed. The snapshots are labeled with, respectively: 'fast; little water', 'less fast; slightly more water', and 'slow; a little bit more water' (C2)

```
Larry           : That it goes slower as there is more in it because it is
                  getting wider
(...)
Unknown student : Because it always goes on like this, of course (student
                  made a growing triangle with his hands, mimicking the
                  cocktail glass' shape). Always more area, it takes longer
                  and longer to fill it
```

Self-evidently, not all students participated in the whole-class discussion. There were, however, no students who expressed disagreement.

Figure 3.10: (Activity 3) Mary, Barbara, and Patricia's model is discrete, non-linear, and coordinates time and height (C2)

During the discussions following activity 3 and 4, the students in C2 used this relationship to explain their models. For example, while discussing Mary, Barbara, and Patricia's third model (Figure 3.10), James argued that the 2.5 second mark should be put higher up. William agreed, because:

```
William   : Yes, that one should go higher, because, that, yes, (?), below
            it goes much faster than at the top
```

After the fourth activity, in which students explored the situation with the computer simulation and made snapshots models, the teacher wondered why the difference of water level height between the first and second snapshot was so much larger than between any two other subsequent snapshots. David answered:

```
David     : Because it is higher and then it's getting wider
```

Again, in both classrooms, students fell back on the relationship between a glass's width and volume.

Conjecture 3 When asked to create a minimalist model, all students would create a discrete model.

All minimalist models (activity 5) were discrete models. Some were table-like, others graph-like, and a couple were a mix of both (as in Figure 3.13). Only one or two student pairs in each classroom incorporated continuous features in their minimalist model by connecting the graphical representations of the discrete snapshots (Figure 3.11).

Conjecture 4 (C1) When two students draw a segmented straight line-graph on the whiteboard, most other students realize that this graph does not fit their continuous understanding of the changing speed, and accept the curve as the better representation.

In C1, the teacher focused the discussion of the minimalist models on the line-graph drawn by Nicholas and Jacob (Figure 3.11). He labeled the axes, redrew the line in red, and together with the students he annotated the axes with numbers (Figure 3.12), conforming the model to common graph conventions known to him and students in primary school. The teacher pointed to a point on the graph (4, 2.75) and asked the students to read it, and he wrote their answer alongside the graph (Figure 3.12). He then guided the discussion towards speed (activity 6), asking if they could find the speed in this graph. Erik argued that this model, a straight line, was incorrect:

```
Eric    : Because it's slanted and it's slanted like that the whole time
Teacher : And that fits with this, with this, this glass?
Eric    : No
Teacher : No, right? And how should it go then, you think? Eric, or Larry
Larry   : I think it should go a bit bent. (Erik draws it in red on top of
          the graph, see Figure 3.12)
(...)
Larry   : In this shape (...) and then, at a certain moment
Teacher : Yes
Larry   : And then, at a certain moment, that it almost doesn't rise any
```

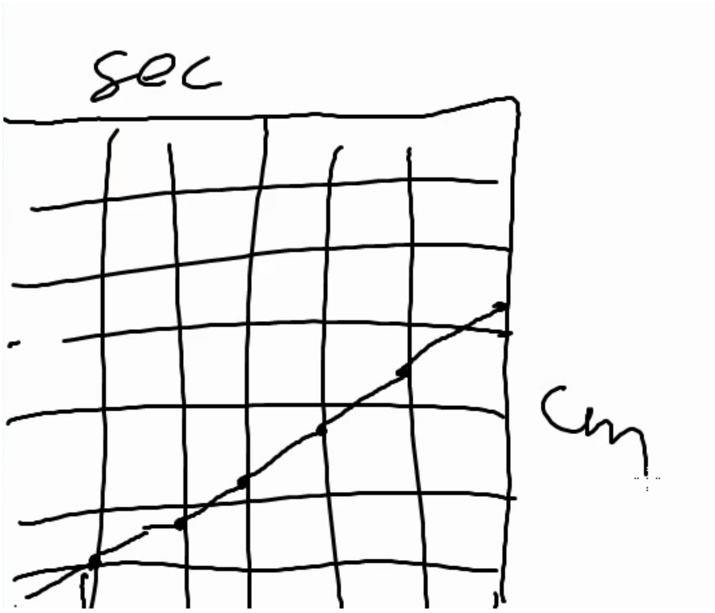


Figure 3.11: Nicholas and Jacob's model at the start of the discussion in C1

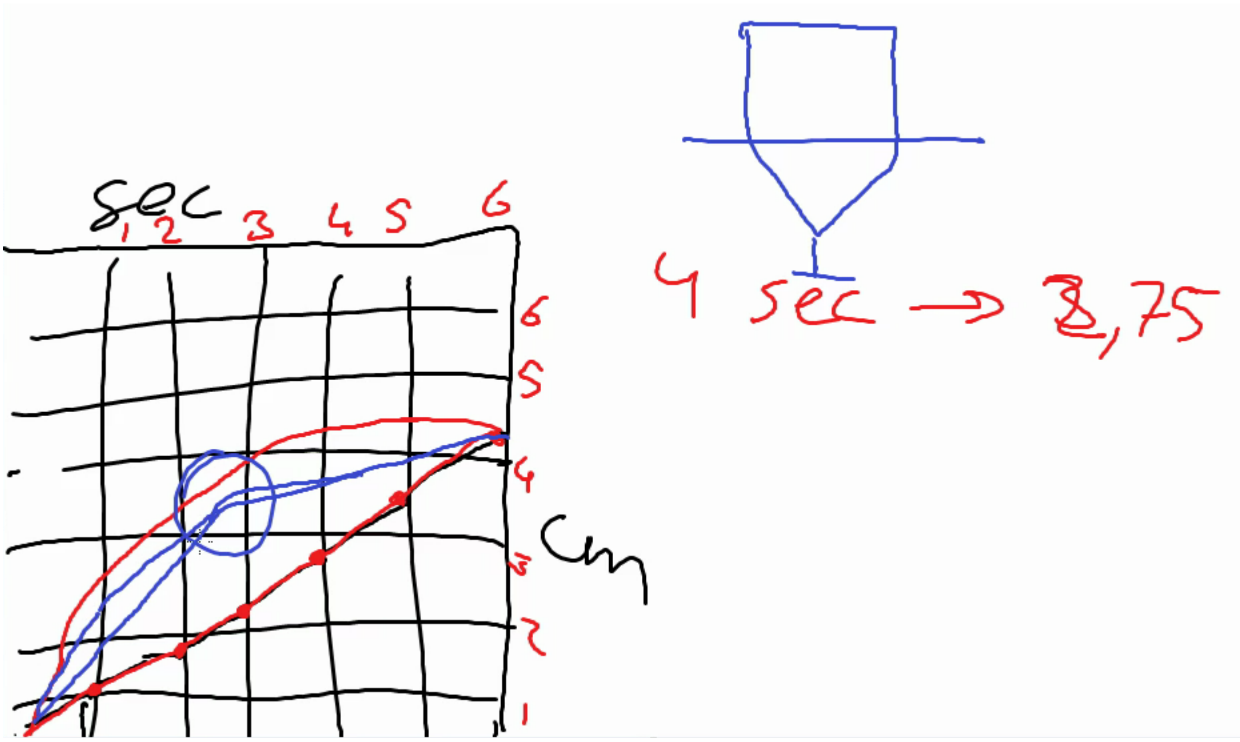


Figure 3.12: Nicholas and Jacob's model at the end of the discussion in C1

more

Eric and Larry immediately saw a mismatch between the graph and their continuous understanding of the situation: because the speed is constantly changing, it cannot be a straight line. When the teacher asked the students to think about filling a wine glass, one student's first solution was to draw two connected straight lines (Fig. 12, blue lines), but in the discussion that followed the students again argued for a curve. However, as only a couple of students explicitly argued for a curve during the discussion, it is unknown to what extent the other students accepted the curve as a fitting model. On the other hand, when the students were asked to graph filling the cognac glass in the third lesson, only one model (of 11) was a straight line; all the others were curves, 9 of which approached the correct curve. This suggests that the students saw the curve as a means to express their understanding of filling glassware.

Conjecture 5 (C2) When the teacher links a student-drawn discrete model to the class's knowledge of segmented line-graphs as a way of representing measurement values, the students start and keep using discrete reasoning until the teacher introduces the continuous graph being generated in the computer simulation.

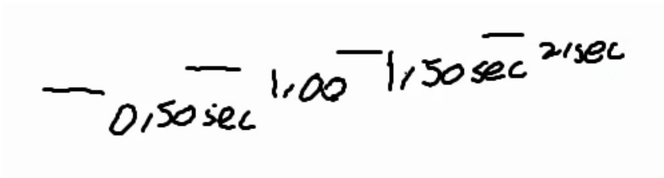


Figure 3.13: David and Thomas's model at the start of the discussion in C2

In C2, the graph-like minimalist model with continuous features was not discussed. Instead, after discussing a number of models, the teacher guided the discussion towards improving David and Thomas's discrete hybrid table-graph model where the water levels were represented graphically by dashes and the times written alongside each dash (Figure 3.13). Patricia suggested to add a ruler to the model (Figure 3.14). While asking the class if they understood this model, the teacher fitted the ruler to the model by adding the minimum and maximum height of the glass's bowl (Figure 3.14). Finally, the teacher had the students identify this model as a graph, emphasizing the representation by sketching a common graph alongside the original model (Figure 3.14, right).

During this discussion, the attention was on individual cases: the representation of number pairs that signified water heights. The students focused on the fragmented

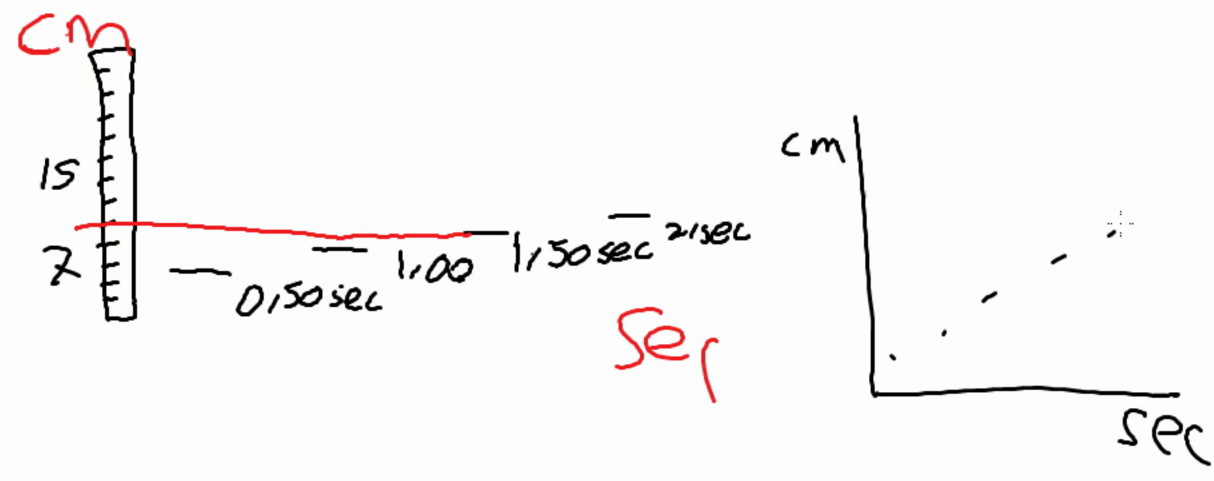


Figure 3.14: David and Thomas's model at the end of the discussion in C2

image of a succession of individual water heights, losing sight of the bigger picture of the whole situation. The students realized that by adding common graph-features, the picture became more precise, but they did not envision the shape of the underlying curve.

Next, the teacher introduced the simulation software by showing the value bars tool and the arrow graph. Although we hoped that by guiding students from their minimalist model to the arrow graph they would come to see the underlying shape, the discussion continued focusing on discrete points and what happened in between two points. However, when the teacher finally introduced the curve, it seemed to fit students' continuous image of the situation. Paul tried to conform it with his image of the situation:

```
Teacher : [With the curve,] You can better see how fast it rises. How can
          you see that?
Paul    : Because with the arrows, you only see a straight line, so then
          it rises everywhere with the same speed. At one, between a part
Teacher : Yes, and is it like that?
Paul    : No, at least not with that shape [of the cocktail glass]
```

Paul realized that the arrows signified that the speed would be constant on every interval in between points, but that did not fit his image of continuously changing speed in this situation.

To Paul, the curve did fit his understanding of the situation. It is unclear to what extent his acceptance of the curve was shared among his peers, but when the students were tasked to graph filling the cognac glass in the third lesson, 8 of 13 models resembled the correct curve; the other graphs were either empty grids or straight lines. This suggests that the curve had become a fitting model to more students than only

Paul.

Conjecture 6 Once the curve was introduced, the students accepted it as a model to describe filling the cocktail glass.

Despite the different learning trajectories in the two classrooms regarding the rediscovery of the curve, once the curve was introduced, the students seemed to accept it. Following conjectures 5 and 6, the curve became a fitting model to the students to describe filling the cocktail glass; they could use it to describe their understanding of filling the cognac glass.

3.4.2 Phase 2: Why did this happen?

Our conjectured learning trajectory (Section 3.3.3.3.4) was based on the assumption that students would start using the snapshots model because they have a discrete conception of speed and have trouble switching from discrete to continuous reasoning on their own. However, that assumption is in contradiction with the actual learning trajectory in C1 where students themselves introduced the curve. To explain what happened, we formulate the following explanatory conjecture:

Conjecture 7 The students come to the classroom with a continuous conception of speed. They only switch to discrete reasoning because of a lack of means for visualizing continuous change.

This conjecture explains what happened in both classrooms. The first part of the conjecture—students come to the classroom with a continuous conception of speed—follows from conjecture 2. Immediately after seeing a cocktail glass fill up, all students realize that the speed is continuously decreasing because of the growing width of the cocktail glass's bowl. Some students, even realized this before they saw the glass fill up.

To further ground conjecture 7 in the data, we focus on three moments during the actual learning trajectory: the first two modeling activities, students' preference of the snapshots model, and the introduction of the continuous curve. If we re-examine the class discussions after the first and second modeling activities in light of this conjecture, students often explain their models, be they realistic drawings or snapshots models, using dynamic verbs like "It fills up" and "The water goes up." The students are describing the situation as a process. Although most students depicted just one moment during that process, given their attention to realistic side-effects of that process, such as bubbling, jumping droplets, and overflowing, it stands to reason that they intended that one moment to stand for the whole process.

Some students tried to capture the continuous changing speed by means of showing changing water levels in subsequent snapshots without much care for the specifics of the snapshots. The snapshots seem to stand for the whole process, not just these particular moments in that process (conjecture 1). When these two types of models were compared, the students expressed a preference for the snapshots model as it better expressed the process than the realistic drawings because, according to Nancy in C1:

```
Nancy    :     You can see how the water goes up
```

Because already 8 students (of 43) used a snapshots model in their first model, it is likely that students were familiar with the idea of using discrete snapshots to visualize change over time. The students lacked the representational competency to model their continuous conception as a continuous model, and opted for the familiar discrete model instead. Unsurprisingly, students started using the snapshots model in the next modeling activities. As a side-effect, the discussions and activities that followed seemed to reinforce thinking about the situation in discrete terms. In the fourth activity, for example, after exploring the situation using the computer simulation, students tried to create more precise models by quantifying time, water level, or volume of subsequent snapshots.

Because there was no discussion about the mismatch between using a discrete model to describe a continuous process, the taken-as-shared notion to use discrete models was not challenged. In this environment, when asked to create a minimalist model, all students created a discrete model (conjecture 3). Only one or two pairs in each group incorporated continuous features in their minimalist model by connecting the graphical representations of the discrete snapshots. Per conjectures 5 and 6, in both classrooms, once the lack of means of visualizing continuous change was remedied by the introduction of the curve, the students connected that continuous model to their prior continuous conception of the situation.

The teacher in C1 kept to the intended learning route regardless of the fact that the students already had introduced the curve, when asked to tell something about the computer-drawn bar graph, the students saw the curve through the bars:

```
Teacher      : You're seeing this [bar] graph, tell something about it
(...)
Nicholas     : May I draw something on it? Can I draw on it [the
               interactive whiteboard]?
Teacher      : Eeh, yes
```

```
Nicholas    : (The students connects the tops of the bars with a line)
              That this is a curve as well, sometimes. This is actually
              a curve as well
(...)
Teacher     : Because, these points [the tops of the bars], what do they
              signify? Ehm, Jessica?
Jessica     : Basically as a line
```

For both Nicholas and Jessica, the discrete bars clearly represented a continuous curve. In both classrooms, this understanding between a point-wise interpretation of a graph and the curve was emphasized by the teacher changing the step-size between points. By a small a step-size, to points clotted together into a line. After the introduction of the curve students never referred back either to the bar graph or the arrow graph in later lessons. This seems to suggest that the curve fit students' initial continuous conceptions of speed rather than that its introduction was a struggle to connect to a discrete conception of speed.

In summary, we may conclude that although discrete representations seem to give rise to discrete reasoning, the underlying conception that the students reasoned from was continuous. In our view, the students reverted to discrete representations and discrete reasoning only for lack of better ways of describing speed. Different from what Castillo-Garsow (2012) suggests, we see no indications of two types of thinkers. They further suggested that "smooth thinking very nearly implies chunky thinking for free" (Castillo-Garsow, 2012, p. 68), which is consistent with how the students in our classrooms effortlessly shuttled back and forth between discrete and continuous reasoning. We want to add that we also found that the setting of the classroom activity does influence how the students reason. Building on a discrete visualization of the filling process, the students in C2 became focused on individual water heights, which obscured the variable rising speed for the students. In C1, where a line graph functioned as a starting point, the students developed an adequate representation of continuously changing speed. Similar to the idea of functional thinking (Vollrath, 1989), to understand instantaneous speed, one has to understand a phenomenon in terms of two co-varying quantities both on a local scale, at individual moments, and on a global scale, as a process of change as a whole.

3.4.3 Abduction on the level of DR: overcoming the conventions of the primary curriculum

The retrospective analysis showed that students' reasoning was grounded in continuous reasoning, while discrete reasoning functioned as a tool to get a handle on continuous processes. It also showed that they easily reasoned about constantly changing speed, which implies that they reasoned about speed at a point, which in turn presupposes a conception of instantaneous speed. This observation triggered a process of abduction, which generated the question: "Do the students ever use average speed?"

A survey of the data showed that average speed was only mentioned in the third lesson while discussing alternative strategies to determine the speed in the cocktail glass. For example, in C2, Charles suggested computing the average speed as a way to determine the speed in the cocktail glass:

```
Charles    : The average, or something?
Teacher    : The average speed of rising? How would that work?
             (Teacher and student together compute the average speed in the
             cocktail glass)
             (...)
             However, can I use this if I want to know what the speed is
             here (the teacher draws a point somewhere on the graph)
Charles    : No, because then you would actually end up with that highball glass
```

Charles suggested to calculate the average speed. After the average speed was calculated, he realized that it could not be used to determine the speed at a moment; only in a highball glass is the speed everywhere the same. In both classrooms the students realized that to determine the instantaneous speed, average speed was no help. Hereafter, average speed was never mentioned again.

The way (Dutch) textbooks are organized suggests that the authors assume that average speed is easier than instantaneous speed. Primary school textbooks introduce average speed and never discuss instantaneous speed. If this assumption of textbook authors is correct one might expect these students to use average speed, which they have been taught before they enter the teaching experiment. This led to the serendipitous conclusion that students do not need average speed to appropriate more formal conceptions of instantaneous speed, and that it would be better to build on their informal understanding of instantaneous speed.

We may even argue that starting with average speed makes the learning process unnecessarily complex. Students often struggle with the notion of average driving speed; they associate 60 km/h with one hour in which 60 km is traversed. Some even believe that you cannot drive at 60 km/h during ten minutes. Stipulating that the students have an informal understanding of instantaneous speed, it makes more sense to build on that and connect it with the concept of constant speed.

3.5 Conclusion and discussion

The finding that the students spontaneously think in terms of instantaneous speed stands in sharp contrast with the common practice of starting instruction on speed by introducing average speed. We may argue that this practice is problematic in that the two meanings of the word speed will easily start to interfere. Furthermore, by focusing on average speed, discrete thinking is promoted, especially given the tendency to use linear situations to start exploring conceptions of change. This could be the source of Castillo-Garsow (2012)'s problematic chunky thinkers. Moreover, building on average speed to understand instantaneous speed brings with it the well-known problems of the limit concept. We therefore argue that one might better start by building on students' informal notion of speed, and try to support students in learning to quantify instantaneous speed, before moving on to average speed. An earlier DE showed that students easily make the connection between the instantaneous rising speed and the constant speed in a highball glass. We believe that this connection can be expanded into a way of quantifying instantaneous speed.

3.5.1 Generalizability

As a caveat we have to note that the generalizability of the findings needs some qualification, because of the uniqueness of the classroom situation. Can the findings be communicatively generalized to be useful (Smaling, 2003) for other classrooms? For the perspective of ecological validity we have to take the specific conditions into account. First of all, only gifted students participated in the study, suggesting a potential plausible transfer to other gifted classrooms. On the other hand, the classes contained a mix of 4^{th} to 6^{th} grade students from different schools and only met twice a week for half a day. In this gifted program, science or mathematics topics were uncommon. The latter might imply that inquiry social norms were well established, but probably not specific socio-mathematical or socio-scientific norms.

As a further limitation to the generalizability, we may note that the teacher did not select the students' models to be presented and discussed in class with an eye on the mathematical agenda. The fact that the students in C1 introduced the line graph seemed more a lucky accident than a controlled act on behalf of the teacher. On the other hand, once the line graph was presented, the teacher did recognize its didactical potential and was able to orchestrate the whole-class discussion in such a manner that the students could deepen their understanding of both the phenomenon and its representation; he spontaneously added the non-linear situation of filling the wine glass as an extra example. In a sense, the teacher was the perfect match for our research:

he is one of few primary school teachers who has followed a calculus course in high school. We cannot expect the average teacher to have as much insight and experience with calculus-related topics and graphs, let alone PCK to teach calculus-like concepts, because these topics are not part of the primary curriculum.

3.5.2 DR and generating new ideas

In this chapter, we set out to illuminate how abduction plays a specific role in DR, a role that is intimately tied to its goal of developing new theory. The latter sets DR apart from (quasi-)experimental research that limits itself to testing theories. In relation to this we may note that scientific progress requires generating theories just as much as testing theories. And Peirce argues that of the three forms of reasoning, deduction, induction and abduction, "abduction (...) is the only logical operation which introduces any new ideas" (Peirce, cited by Fann (1970), p.10). Where deduction and induction are the foundation of experimental research, abduction and DR are interdependent. On the one hand, abduction is essential for generating new ideas, while on the other hand, the iterative nature of DR and its focus on "understanding the messiness of real-world practice" (Barab & Squire, 2004, p. 3) invites the unexpected to happen, which fuels abduction.

We highlighted how new ideas were introduced at two different levels of research. First, abductive reasoning played a role when the researchers thought up adaptations to the conjectured LIT in a micro-design cycle—which were explored further in the subsequent micro-design cycles. Second, at the level of the macro-design cycles, abductive reasoning played a role in generating new ideas in service of the overall aim of the DR project. At both levels, however, abductive reasoning was triggered by unexpected events, either during a micro-cycle or a macro-cycle, and resulted in a new piece of the LIT.

As we mentioned in the introduction, abduction may be linked to Smaling (1992)'s conception of methodological objectivity and its two requirements, to avoid distortion and to let the object reveal itself. Typical for DR is its aim to do both. Whereas in quantitative research, doing justice to the object of research is primarily translated as avoiding distortion by requiring reliability, and the common conception of validity is only distantly akin to letting the object reveal itself. The classic conception of validity surely does not entail the openness that Smaling's methodological norm asks for. In relation to this we want to stress the importance of openness to "listen" to the object of research in DR. Here the researcher may take an active role by trying to exchange his or her observer's point of view for the actor's point of view (Figueiredo, van Galen, & Gravemeijer, 2009) of the students. Or to put it differently, DR requires genuine, concerted interest in student thinking. Surprising events may be helpful in this respect, in that they are an indication that there is a clear misalignment between researchers' prior understanding of students' learning processes and the actual learning processes. To resolve this distortion, the researchers have to re-examine their prior conceptions while considering students' perspective more strongly and formulate new explanatory conjectures. Subsequently, if these new conjectures can be grounded in the data collected, and are added the LIT, the LIT itself becomes more objective as it does do better justice to the students' learning process.

Admittedly, abduction does not offer the same rigor as deduction and induction. However, abduction is very similar to what Gould (2011) calls "consilience," the way of validating theories which he puts forward as representative for the more holistic approach of the humanities. He describes this as "the validation of a theory by the 'jumping together' of otherwise disparate facts into a unitary explanation" (ibid., p. 192). In our example, the assumption that students come to the classroom with a continuous conception of speed may fulfill a similar function, as it explains that: the students started talking about speed that is changing constantly when they saw the cocktail glass filling up they effortlessly accepted the curve as an adequate description of this phenomenon students in general struggle with the concept of average speed.

Further, we have to keep in mind that the primary goal of DR is to find out how things work, and not to establish for a fact how things are. In closing, we may conclude that abductive reasoning is typical for DR and fits with the aim of understanding and uncovering process-oriented causality (Maxwell, 2004).

4

Design research as an augmented form of educational design: Teaching instantaneous speed in fifth grade*

4.1 Introduction

Design research emerged in the 1990s as a reaction to a perceived gap between educational practice and research (The Design-Based Research Collective, 2003; Reeves et al., 2010; van Eerde, 2013). To bridge that gap, design research builds on educational design to take real-world educational practice into account while using a process of iterative refinement to create both a product and a theory that explains how that product works in terms of students' learning processes (Plomp, 2013; Cobb, 2003;

*This chapter is based on Beer, H. de, Gravemeijer, K., & Eijck, M. van. (submitted) Design research as an augmented form of educational design: Teaching instantaneous speed in fifth grade.

Barab & Squire, 2004). However, most products of design research do not evolve beyond mere prototypes (Burkhardt & Schoenfeld, 2003; Schoenfeld, 2009), which could potentially limit the utility of the accompanying theory. From an educational design perspective, Schoenfeld (2009) attributes this problem to the fact that many design researchers do not seem to understand the need for both theoretical development *and* practical design. As a solution, he proposed that design research is to be carried out in balanced project teams (Schoenfeld, 2009). However, this might not be a realistic option for many a design research project. Since design research is ultimately used in research settings, the proportion of research expertise in project teams often exceeds that of educational design expertise. It would further suggest researchers have to acquire the necessarily educational design skills somehow.

Unfortunately, one can only learn to become an educational designer through an inefficient process of experimentation and apprenticeship because educational design lacks both an institutionalized form of schooling and professional literature (Schunn, 2008). To professionalize educational design practice, Schunn (2008) suggests to model it after the field of engineering, which has a long standing tradition of scholarship and education on design practices. Schoenfeld (2009) continues this argument and proposes that the educational design community starts documenting their practices and knowledge. We concur, and we see a role for design research in this regard. Design research may take various forms, the type of design research we have in mind here focuses on learning processes (Gravemeijer & Cobb, 2013). In this type of design research, special attention is paid to the origin and development of the theoretical claims, in a process, which to a large extent comes down to documenting both the actual educational-design practice and the knowledge construction. In this sense, design research may offer indications for a similar approach in educational design.

Design research does not follow a (quasi-)experimental research design. Instead, the process by which claims are produced will have to justify its results in a more direct manner. In relation to this, Smaling (1990) borrows "virtual repeatability" from ethnography as an alternative for the prevailing criterion of repeatability. He links this methodological norm of virtual repeatability to the criterion of "trackability". Outsiders should be able to retrace the learning process of the researchers in such a manner that it becomes virtually repeatable for them. This then would allow them to judge the credibility of the claims themselves.

Characteristics for design research with a focus on learning processes is that the design revolves around the mental activities of the students. Here we may think of Simon's (1995) notion of a hypothetical learning trajectory, which describes the anticipated mental activities the students will engage in when they participate in the envisioned instructional activities, and how those mental activities relate to the learning goals. It is the design, trial, evaluation, and revision of a series of such hypothetical learning trajectories that is at the heart of the learning process of the designers. The consecutive hypothetical learning trajectories are reflexively related to the so-called local instruction theory. The local instruction theory constitutes the rationale for the instructional sequence that is being developed and consists of a theory about a possible learning process for a given topic, and the means of supporting that process. That is, the researchers start out with a conjectured LIT, which is revised and adapted during multiple design experiments.

In this chapter we describe such a learning process to illuminate how design research may contribute to the codification of educational design practices. While at the same time trying to show that design research can produce theory that is relevant for educational designers.

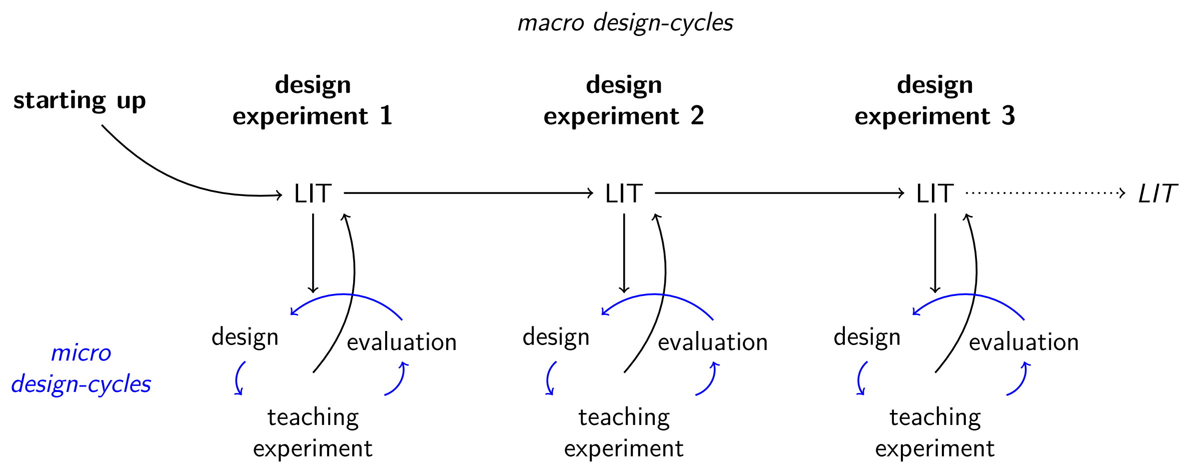


Figure 4.1: Schematic overview of design research aiming at developing a LIT

In the design research project we discuss here, we followed the approach outlined in Gravemeijer and Cobb (2013) in order to develop a local instruction theory (LIT) on teaching instantaneous speed in 5th grade. This LIT, and the prototypical instructional sequence by which it was instantiated, was elaborated, adapted, and refined in three macro design-cycles called design experiments (Cobb, 2003; Gravemeijer & Cobb, 2013) (Figure 4.1). A design experiment has three phases. In the preparatory phase, the LIT is (re)formulated and used to design an instructional sequence. Subsequently, that instructional sequence is tried in a teaching experiment that consists of a sequence of micro design-cycles of (re)developing, trying out, evaluating, and adapting instructional activities and materials (Figure 4.1, blue cycles). Next, a retrospective analysis is carried out. This analysis of what happened during the teaching experiment informs a second round of analysis of how the design affected students' learning processes. The results are used to refine the LIT, which acts as the starting point of the next design experiment. Before the first design experiment is conducted, design research is started up by elaborating the potential learning goals, and exploring the literature and any other sources that might contribute to the researchers' understanding of students' prior conceptions.

In the next section, we explore the development of the instructional sequence and the corresponding LIT and its justification by describing our own learning process during the development of an instructional sequence on instantaneous speed in our design research. Next, we elaborate this justification further by detailing its role in pertaining to the trackability of our research. Finally, we discuss these justifications in light of the relationship between educational design and design research.

4.2 Interwoven development of the LIT and our own learning process

In this section we describe the development of the LIT to illuminate our own learning process during the development of an instructional sequence on instantaneous speed in our design research project. As it is almost impossible to give a full and complete account of our learning process, we focus on the main learning moments and we will briefly summarize what we learned after each design experiment.

4.2.1 Forming our initial ideas

This section is summary of the results and findings presented earlier in Chapter 2.

Starting up

We started by reviewing the literature on primary-school students' conceptions of speed. We found that both in this literature and in the primary curriculum speed is mostly limited to average speed in the context of motion and described by a ratio; instantaneous speed is not mentioned. Instantaneous speed is conventionally first taught in a calculus course, we therefore widened our review to the literature on teaching calculus-like topics earlier in the mathematics curriculum than calculus is conventionally taught. Most of these initiatives used computer simulations and graphs: it enables students without much algebraic skills to reason about change (Kaput & Schorr, 2007; Stroup, 2002). Exemplary is Stroup's (2002) "qualitative calculus" approach to teaching calculus-like topics. Stroup critiques the conventional instruction of rate, which treats intuitive qualitative understanding of rate merely as a transitional phase towards a ratio-based understanding of rate, and he argues developing a qualitative understanding of rate is a worthwhile enterprise in and of itself. We decided to aim at expanding on this approach by supporting 5th graders in developing both a qualitative and a quantitative understanding of rate.

In our search to develop a conception of instantaneous speed while circumventing the complex process of taking the limit of average speeds we looked at the historical development of the concept of instantaneous speed. Long before a formal mathematical definition, an intuitive notion already existed (Doorman, 2005): around 1335, William Heytesbury defined his intuitive notion of instantaneous velocity as:

> 'a nonuniform or instantaneous velocity (...) is not measured by the distance traversed, but by the distance which would be traversed by such a point, if it were moved uniformly over such or such a period of time at that degree of velocity with which it is moved in that assigned instant.' (From: Heytesbury (1335) *Regule solvendi sophismata*, as cited in (Clagett, 1959, pp. 235-237)).

We assumed that the students' intuitive notions are close to this historical notion. We therefore formulated a LIT on teaching instantaneous speed in 5th grade, in which we aimed at helping the students to explicate and expand these notions.

We chose the context of filling glassware over the motion context, as we feared that the latter might bring with it unreflected terms and procedures. The origin of this context can be traced back to the experienced educational designer Malcolm Swan, who used it in a secondary textbook on functions and graphs (Swan, 1985). We conjectured that the students would come to realize that the speed with which the water level rises in a cocktail glass changes at every instant, and that this speed is continuously diminishing. Building on this insight, the students would come to understand how the shape of a "height-over-time" graph relates to the rising speed in the glass. By discussing the curved shape of this graph we expected that the students could be made aware that the speed changes at every instant. By making the latter a topic of discussion, we would have a starting point for deepening their understanding of instantaneous speed.

Preliminary one-on-one teaching experiments

Before elaborating those ideas further, we carried out a series of one-on-one teaching experiments with nine 5th graders—which we reported on in detail in Chapter 2. We developed a short instructional sequence with the same activities with a highball glass (a cylindrical glass), a cocktail glass, and an Erlenmeyer flask in a computer simulation. For each glass, the students were asked to:

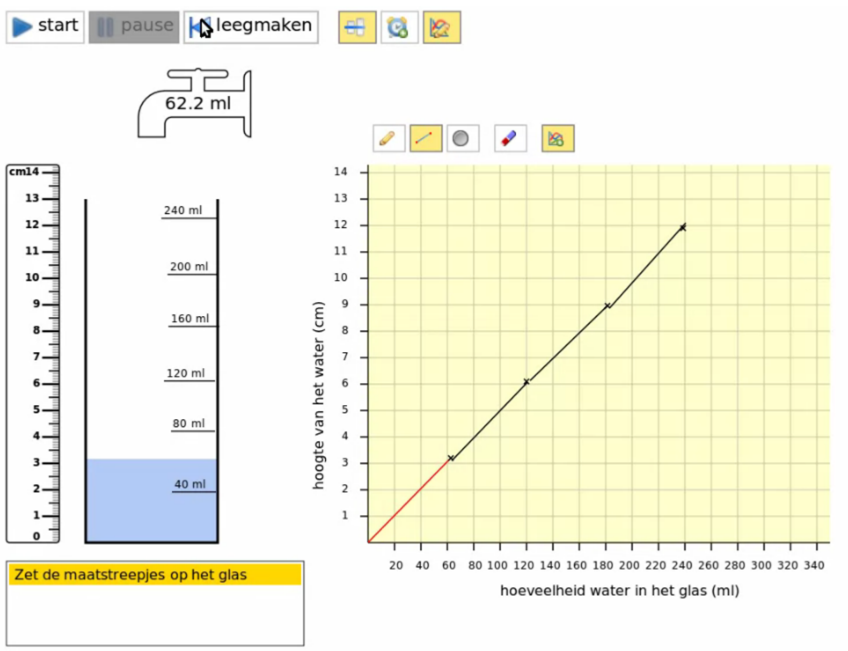


Figure 4.2: Computer simulation of filling a highball glass. (Click to explore)

- turn the glass into a measuring cup by dragging hash marks to the right place on the glass (see Figure 2 and 3, left hand side),
- evaluate their solution using the computer simulation,
- draw a graph of filling that glass, and
- again, evaluate their solution using the simulation (see Figure 2 and 3, right hand side).

We performed 8 one-on-one teaching experiments with above average performing 5th graders (in one experiment a pair of students participated).

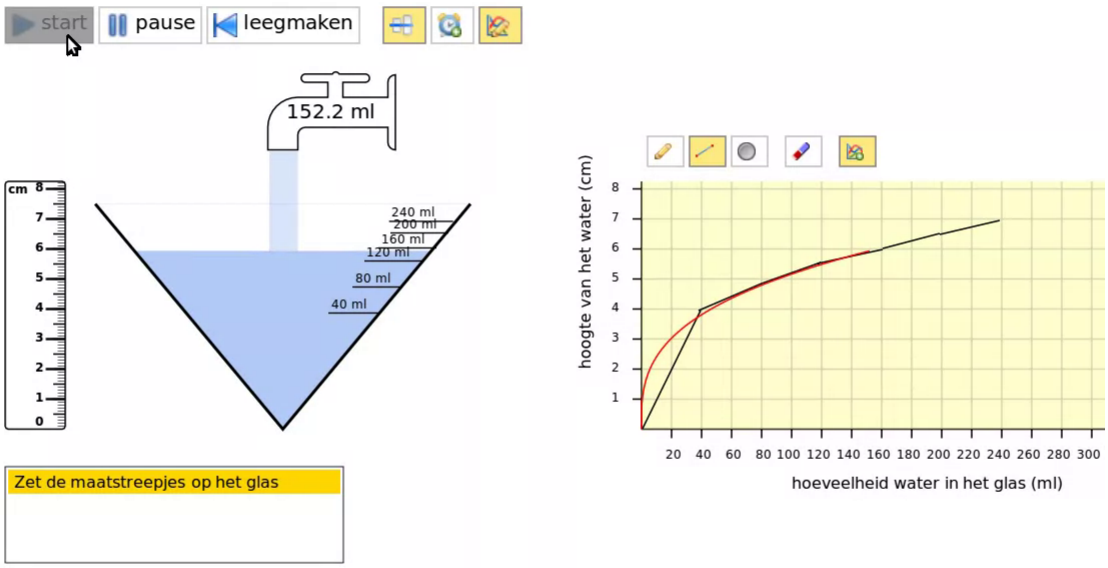


Figure 4.3: Computer simulation of filling a cocktail glass. (Click to explore)

In the one-on-one experiments, it showed that the students found the linearity of the process of filling a highball glass self-evident, and spontaneously described this process with a linear graph (Figure 4.2). When carrying out those tasks for a cocktail glass, however, many students initially fell for the so-called linearity illusion (de Bock et al., 2002). Most students made a linear-gauged measuring cup of the cocktail glass. But when they saw a cocktail glass fill up, they quickly realized that the process was not linear. They could relate the decreasing speed to the glass' increasing width, arguing that the speed decreases because of the growing width of the glass. In line with this realization, the students accepted the computer-drawn curve as a better graph for describing filling the cocktail glass than their own hand-drawn straight-lined graphs. They could not, however, explain the curve. Nor could they come up with such a curve by themselves.

We may briefly summarize what we learned:

- Conceiving filling a (cylindrical) highball glass as a linear process appeared self-evident for the students.
- This was also the case for the linearity of the graph.
- With the cocktail glass they initially fell for the linearity illusion, but when they saw the simulation, they realized that the rising water level slows down.
- They could not draw the correct graph.

- They seemed to appreciate the curve, but could not explain it.

4.2.2 Design experiment 1

In the first lesson of the teaching experiments of design experiment 1 we tried to imitate the starting-up phase, by asking the students to make measuring cups from the highball glass, the cocktail glass, and the Erlenmeyer flask, and evaluate their work by filling up the glasses in the computer simulation.

Based on the one-on-one teaching experiments, we expected the students to have no trouble with the highball glass and apply the same linear prototype to the cocktail glass. Once they would observe it fill up, they could realize it is a non-linear situation. The students would relate the decreasing speed to the glass' increasing width, while formulating the relationship between a glass' width and speed. Hereafter, the students would be able to coordinate water height and volume.

Next, the students would be asked when the water rises with the same speed in both the highball and cocktail glass. Given their informal understanding of the relationship between a glass' width and speed, they would be expected to come up with the point where both glasses have the same width. We then wanted to support them in realizing they can compute the speed at a specific point by computing the corresponding virtual highball glass' speed. When prompted to think of a way to determine the speed at another point, we expected the students to imagine a highball glass with the same width as the cocktail glass at that point, and compute its speed (see Figure 4.4). After introducing a computer-tool to draw an imaginary highball glass on top of other glasses, the students would have a tool available for measuring instantaneous speed in various glassware.

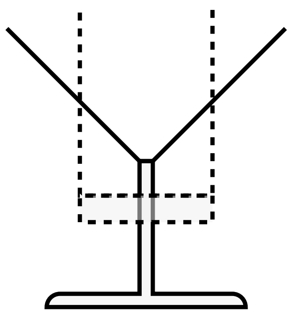


Figure 4.4: Using constant speed in an imaginary highball glass to quantify instantaneous speed in a cocktail glass

Then, the students would start exploring the changing speed quantitatively by measuring instantaneous speed at various points. We initially planned to deepen their emerging quantitative understanding by having them draw speed-volume graphs. Followed by tasks to reconstruct the shape of a glass on the basis of a glass' speed-volume graph. However, when trying out this learning trajectory in a mixed 5th/6th grade gifted classroom with an experienced teacher, we observed in the third lesson that the students seemed to become bored with filling glasses. As we had planned one-on-one teaching experiments in between lessons four and five to explore students' conceptions of speed in other contexts, we decided to alter the fourth lesson by including these activities about other contexts: toy-car racing and daily temperature (see the last four activities of lesson 4).

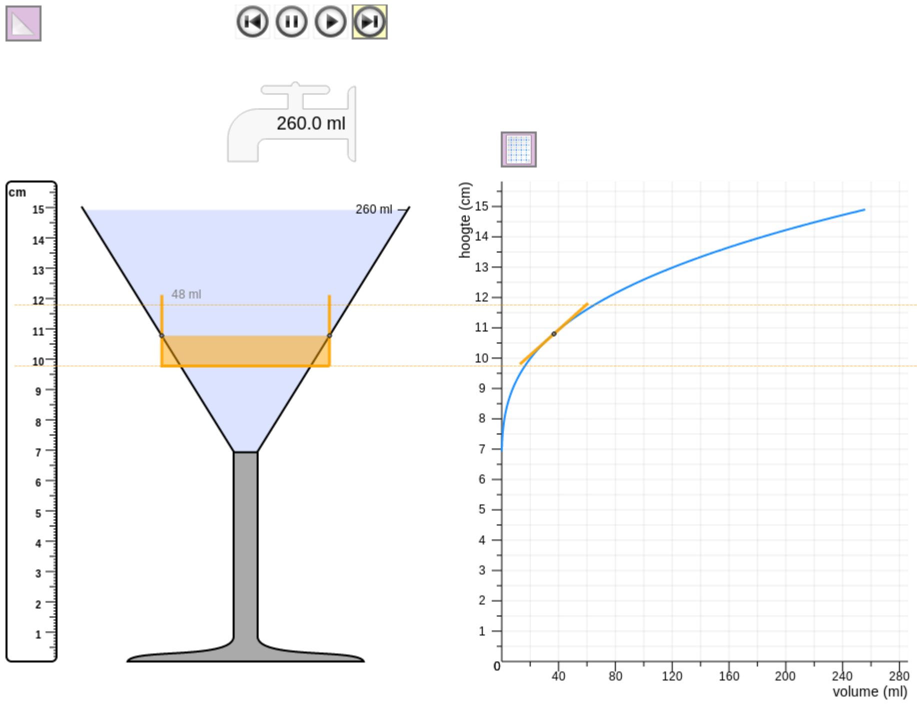


Figure 4.5: Linking the speed in an imaginary highball glass to a tangent line. (Click to explore, choose "Klassikaal 1" and press the "triangle" button in the top-left corner)

We realized, however, that to determine speeds in those contexts the students had to be able to work with the tangent line on a curve. We conjectured that by combining the highball-glass-tool and the tangent-line-tool, the students would come to see that the tangent line in a point of the graph corresponds with the linear graph of the speed in the virtual highball glass (Figure 4.5). Then we expected the students to be able to use the tangent-line as an indicator of the speed in a point of a graph even when the graph is shown without the realistic simulation.

Observations, reflections, and revisions

The first instructional activities were intended to lead the students to coordinate two quantities, but, as a side effect, it made the students think about change in terms of hash marks, height differences, and intervals. This discrete approach to change was being reinforced by the way the teacher approached whole-class discussions about computing speed in the highball glass: she wanted the students to mark measurements from a table as points of the graph first, and then connect the points with straight lines even when some students argued to draw a straight line immediately. She led the class to focus on determining speed on small intervals, without discussing what a computed speed actually meant. Thus, when the highball glass was replaced by the cocktail glass, the students interpreted this non-linear situation in terms of a series of discrete changes. They did have trouble matching that perspective with the continuous changing speed in the cocktail glass.

It showed, however, that the students had no problem whatsoever with the question when the speeds in the cocktail glass and highball glass are the same. Indeed, already at the end of the first lesson the students realized that this would be when the widths were the same. As anticipated, this allowed the students to invent the virtual highball glass as a measure for instantaneous speed. Linking the highball-glass-tool directly with the tangent-line-tool in the computer simulation (see Figure 4.5), seemed to enable the students to determine instantaneous speeds in the cocktail glass by using the tangent-line-tool, although the extent of their understanding of the tangent line and its relation with speed remained unclear.

In the context of toy-car racing, the students were able to interpret the graph, and most could compute the instantaneous speed using the tangent-line-tool. However, the concept of instantaneous speed was not discussed. And we suspect that the students' skill in using these tools to compute instantaneous speed was mostly procedural in nature, and not an expression of a deeper understanding of instantaneous speed.

Chunky thinking During the first three lessons, the students displayed what Castillo-Garsow et al. (n.d.) denotes a chunky way of thinking, which was also promoted by both instructional sequence and teacher. Castillo-Garsow (2012) introduced the concepts "chunky" and "smooth" to characterize two different forms of reasoning. Thinking about change in terms of intervals, or completed chunks, is called "chunky". Students with a chunky image of change see change on an interval as the end-result of change on that interval. Students that see change as a continuous process, however, have a "smooth" image of change (Castillo-Garsow, 2012; Castillo-Garsow et al., n.d.). Smooth thinking is fundamentally different from chunky thinking, and chunky thinkers will have trouble thinking about change as a continuous process. Contrariwise, thinking in terms of chunks, however, is apparently relatively easy achievable from a smooth perspective (Castillo-Garsow et al., n.d.).

We may briefly summarize what we learned:

- We became even more aware that the power of the context of filling glassware lies in the fact that it offers the students a powerful theory to reason about the covariation process on the basis of their understanding of the relationship between a glass' width and its speed.
- The students showed their implicit conception of instantaneous speed, when answering the question, "When is the speed in the cocktail glass equal to the speed in the highball glass?": for this answer presupposes that one thinks of speed in a point.
- The students could handle the tangent-line-tool, but we doubted their level of understanding.
- Further, they still did not manage to come up with a continuous graph; we believe that starting with the highball glass might have put them on the wrong track.

4.2.3 Design experiment 2

The aforementioned chunky thinking is highly problematic because developing and reasoning about a smooth graph is a key element in the envisioned learning process. To overcome the problem of a discrete learning environment reinforcing discrete thinking, we decided to change the LIT for design experiment 2 to let the students' learning process revolve around the non-linear situation of filling the cocktail glass. In this we followed Stroup (2002), who argues that linear situations might be too simple to support students developing understanding of calculus-like topics. We also removed the measuring cup activities because these invited students to focus on interpreting a continuous process in a discrete manner.

Modeling-based learning

To increase conceptual discussion about speed we opted for a modeling-based learning approach. Modeling is generally seen as the core activity in science, and a natural element of STEM education. A modeling-based learning approach aims at getting access to the students' thinking by having them express their understanding in visual or "expressed" models (Coll, France, & Taylor, 2005; J. Gilbert & Boulter, 1998). An expressed model can be presented, discussed, and evaluated in class, allowing students to refine their models, while giving the teacher (and researchers) indirect access to their thinking. Besides, through this modeling process, a consensus model may come to the fore, which may become taken-as-shared in the classroom.

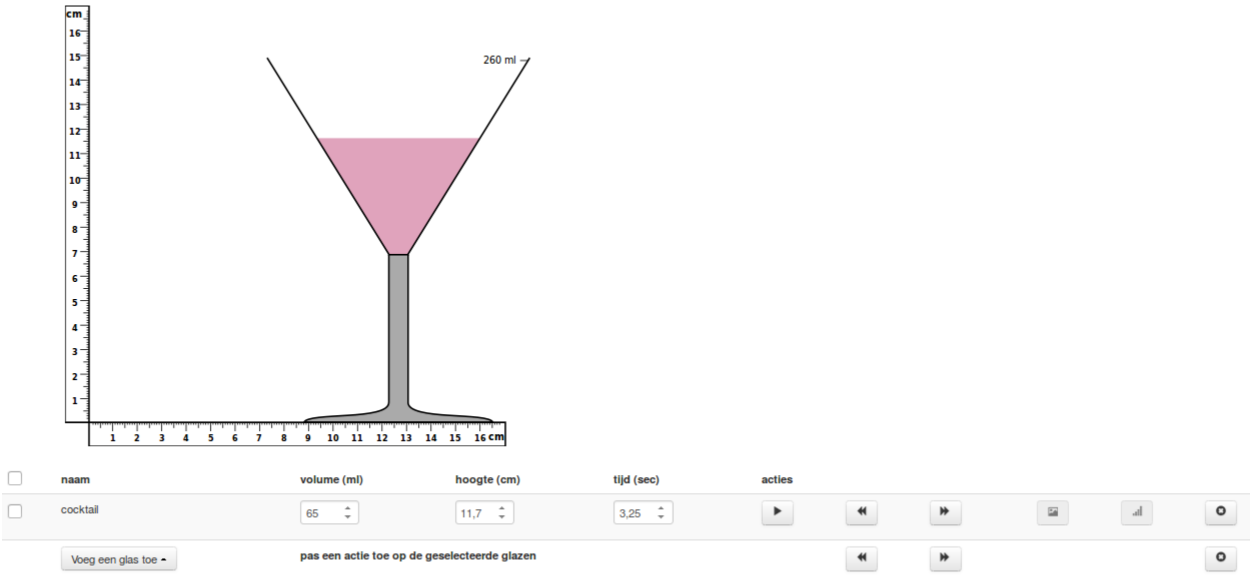


Figure 4.6: Computer simulation of filling the cocktail glass: explore volume, height, and time. (Click to explore)

We redesigned the conjectured learning trajectory with a modeling-based learning approach in mind: We would ask the students several times to model the speed with which a cocktail glass fill ups, and to improve their model. We expected the students' first models to be quite realistic depictions of the situation (Schwarz, 2009), which would be the starting point to support the students in developing more mathematical models. When these first models would be evaluated by running the computer simulation (Figure 4.6), we expected the students to be surprised by the non-linear nature of the situation, but immediately realize why the glass fills up the way it does. When asked to make a new model, we expected them to express the non-linear nature of the situation by coordinating water height and time. Next, in order to shift the students' attention toward the quantitative coordination of water height and time, we would ask them to create a model to predict the water height at certain moments. We expected them to create more precise models by using the simulation to measure the water height and time and use these measurements to annotate their models.

Although students were to create more precise models, we did not expect them to start using Cartesian graphs on their own. On the other hand, van Galen et al. (2012) showed that children at this age are able to reinvent graphs when supported by a suitable instructional sequence. Once the Cartesian graph was introduced, however, we did expect the students to value it as a useful model. With initial guidance from the teacher, the students could use the graph to predict the water height at any moment. Furthermore, even though they were familiar with straight-lined graphs, once the (computer drawn) graph was introduced we expected the students to connect its shape to their image of filling glassware. They would extend their understanding of the relationship between a glass' width and speed to include the steepness of the curve: the smaller a glass, the faster the water rises, the steeper the curve. They would realize that the speed changes continuously because the glass' width changes continuously. However, we did not expect them to be able to determine the speed at a given point.

Then we would shift to a brief exploration of the speed in a highball glass. We expected students to be able to depict filling the highball glass, to explain why it is a straight line, and to compute the constant speed. Next, the students would be guided towards construing the highball glass as a tool for measuring instantaneous speed by exploring when the speed is the same in both glasses.

Building on the realization that when the two glasses have the same width, they would have the same speed, the students were expected to come to see that the cocktail glass' curve is as steep as the highball glass' straight line at that point. Finally, we explored if the students could use this understanding in the context of toy-car racing. We expected the students to explain a race given a graph, indicate where the car went fastest or slowest, and use the tangent line to quantify its speed.

Observations, reflections, and revisions

Modeling-based learning We tried this conjectured learning trajectory out in a small-scale teaching experiment with four 5th graders; the first author acted as a teacher. The modeling-based learning approach was a success with respect to generating more conceptual discussion about speed. Modeling helped students and teacher alike to express and discuss their thinking in more detail. There was ample room to explore alternatives. The students had some difficulty with combining the graphs of the cocktail glass and the highball glass. Eventually, they figured out that they had to draw a line through the appropriate point of the cocktail-glass graph, parallel to the highball-glass graph. This offered a basis for introducing the tangent line, and most students were able to determine the instantaneous speeds in the cocktail glass by using the tangent line.

In this manner, the students developed a deeper qualitative understanding, however, their quantitative understanding of instantaneous speed lagged behind. Although they were able to use numerical procedures to compute speed, their ability in applying these procedures was limited. Students' prior understanding of speed as a ratio of distance over time appeared less well developed than anticipated. Furthermore, students' command of units and quantities was touch-and-go. We concluded that as primary-school students' prior understanding is underdeveloped, we would have to pay more explicit attention to units, quantities, and computing speed as part of the learning trajectory.

Students appear inherently chunky thinkers The other main change we had made to the LIT after design experiment 1 did not pan out as we had hoped. We had conjectured that starting the exploration of speed with the cocktail glass and skipping the measuring cup activities would steer students away from discrete thinking and make it easier to interpret filling glassware as a continuous process. However, despite the fact that the students seemed to realize that the speed changed continuously, their models still reflected a discrete perspective. And when the students tried to incorporate more quantitative aspects, their models became even more discrete. This appeared to fit Castillo-Garsow (2012)'s characterization of chunky thinkers. We therefore conjectured that students of this age might be chunky thinkers who have to be supported to in making the switch from discrete to continuous thinking about speed (Castillo-Garsow, 2012).

We may briefly summarize what we learned:

- The modeling activity does help to make the students' thinking visible and topic of discussion.
- The students show lack of understanding of, and fluency with, measures of speed.
- Still no success in developing continuous graphs: we conjecture that maybe students of this age are chunky thinkers.
- However, the students construed the tangent line to the cocktail glass' curve parallel to the graph of the highball glass as an indicator of the speed in a given point.

4.2.4 Design experiment 3

This section, in particular the vignette, is partially based on the results presented in Chapter 3.

Working towards continuous graphs

We decided to start again with asking the students to explore the glass-filling context by repeatedly modeling how the cocktail glass fills up. Then, instead of just introducing the curve, we added two activities to support students in developing the continuous graph by building on their own models. First, the students would be asked to draw on paper a minimalist model by taking one of their previous models as a basis and removing those characteristics that were not absolutely necessary to describe the situation. We expected them to make a step towards more abstract discrete models by removing realistic elements such as glasses and water sources. Then, based on these minimalist models, the teacher would guide the students to adapt their models to become increasingly graph-like representations, starting with a bar graph, followed by an arrow graph, leading up to a continuous Cartesian graph. The bars in the bar graph would come to signify water heights at specific moments, and arrows connecting these points would come to signify change (Figure 4.7). In this phase, we expected the graph—as a model—to still derive its meaning for the students from its reference to the actual situation of filling the cocktail glass. Reflecting on computer-drawn continuous graphs and student-generated graphs in relation to the shape of the glass, we expected the students to come to see the curve as signifying both the changing value and the instantaneous speed.

Quantifying speed

Following the introduction of the curve, we would continue with the exploration of quantification of instantaneous speed by introducing the highball glass and asking the students to draw its graph on paper. Here, the teacher would activate the students' prior knowledge of computing speed by explicitly referring to familiar notions of speed as distance over time before moving towards computing the rising speed in the highball glass. By paying attention to the quantities and the units involved, we expected the students to become better aware of what speed means in this situation. Through exploring speeds in various highball glasses, the students would become familiar with computing speed.

Building on the students' understanding of the relationship between a glass' width

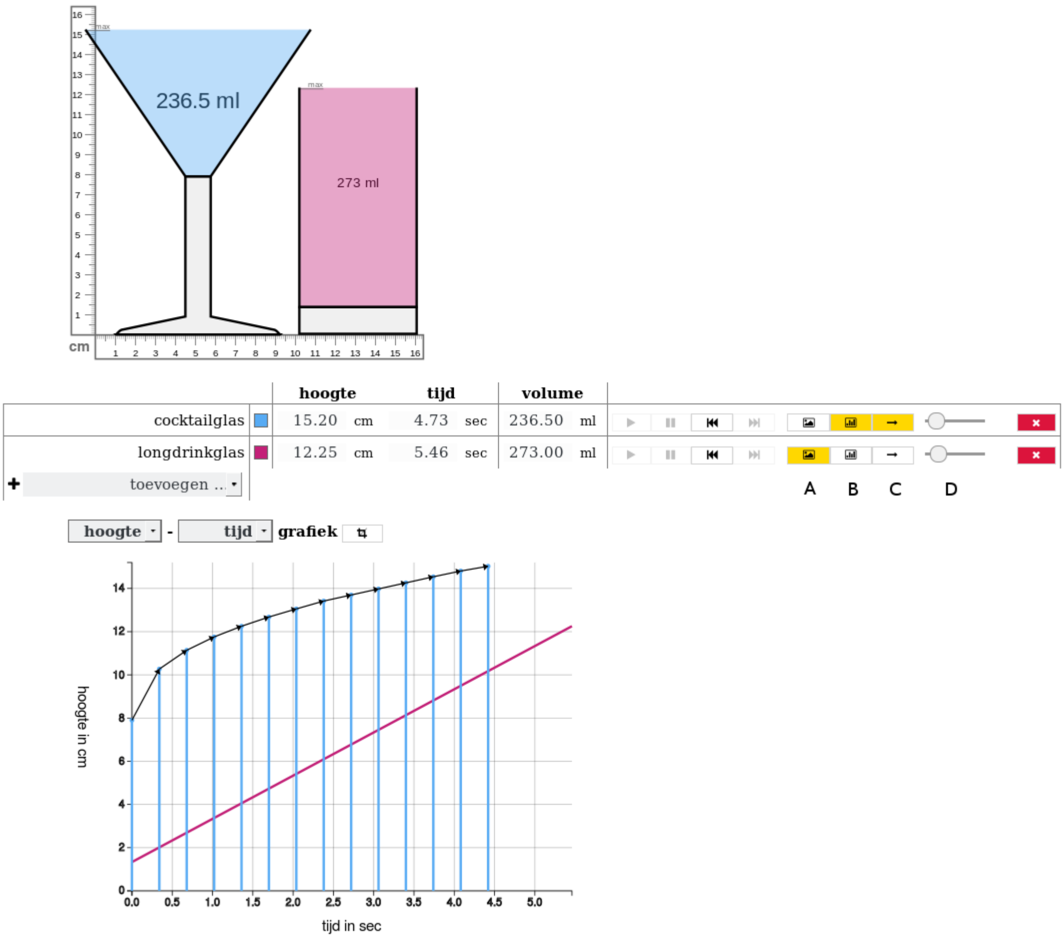


Figure 4.7: Simulation software showing graphs of the cocktail glass (green) and the highball glass (purple). Three different kinds of graph can be toggled by buttons A, B, and C, respectively the bar graph, the arrow graph, and the line graph. With slider D the interval between the bars and the arrows can be changed, showing more or fewer points. (Click to explore)

and speed, they would be guided to invent the highball glass as a tool for determining the instantaneous speed in the cocktail glass. Next, this understanding would be expanded by linking the graph of the highball glass to the steepness of the curve of the cocktail glass at the point where both glasses have the same width. Then the tangent-line-tool would be introduced as a means for measuring instantaneous speed (Figure 4.8).

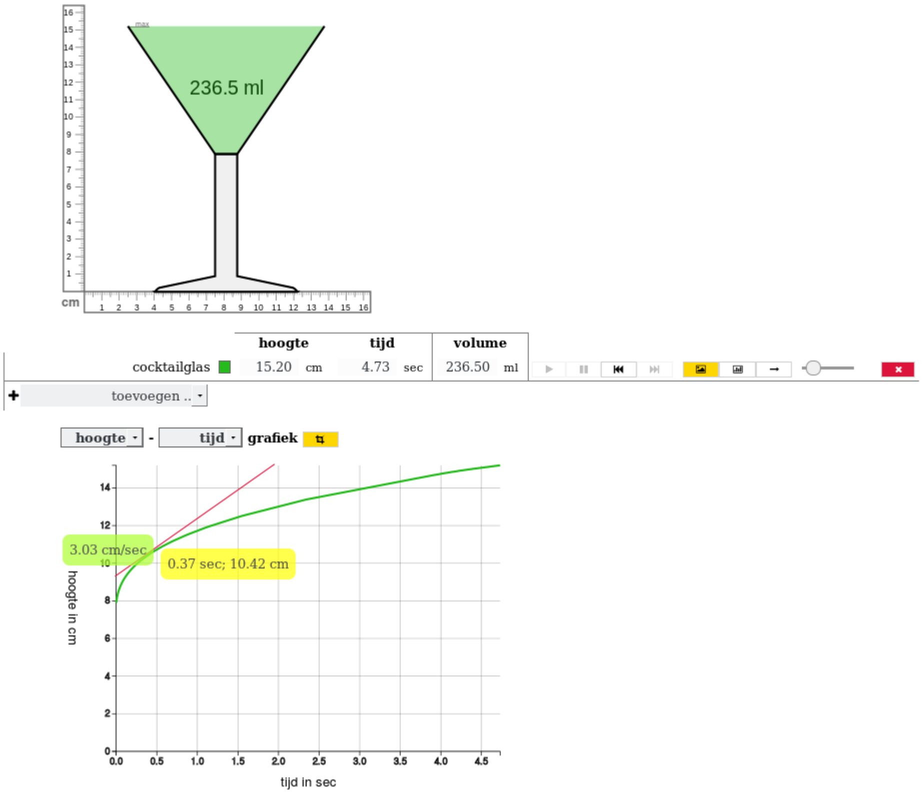


Figure 4.8: The tangent-line-tool for measuring instantaneous speed. (Click to explore)

Next, we would switch to the familiar context of cooling down and heating up of water. The students would be asked to graph what happens when a glass of hot water cools down or a glass of ice-water heats up. After discussing these graphs, we would ask students to repeatedly measure the temperature at several moments and to compute the differences with the previous measurement and the room temperature, followed by computing the cooling speed and predicting when the cooling process will be finished. We expected them to improve their first models and to be able to point out where the temperature of the water cools down the fastest and why that is. Thus, we expected them to reinvent a qualitative version of Newton's law of cooling.

Observations, reflections, and revisions

Students constructing continuous graphs We translated the conjectured learning trajectory into an instructional sequence and tried it in two mixed 4^{th}-6^{th} grade gifted classrooms taught by the same teacher. A month after the first three lessons about filling glassware, a fourth lesson was taught about cooling and heating. Each lesson was first taught in classroom 1 (C1) on Fridays and then, after the weekend, in classroom 2 (C2).

The first lesson went in a similar manner as in design experiment 2: the students created increasingly more discrete models. When the students were asked to create a minimalist model with only the necessary elements to describe the situation in the next lesson, there was one pair of students in each classroom that created a graph-like model.

In C2, the students did need the guidance of the teacher—as we had anticipated. Building on a discrete visualization of the filling process, the students in C2 became focused on individual water heights—which obscured the bigger picture of how the rising speed changed over time. In C1, however—where a segmented-line graph functioned as a starting point—the students developed an adequate representation of continuously changing speed as an improvement of a given line graph, as shown by the following vignette:

start of vignette

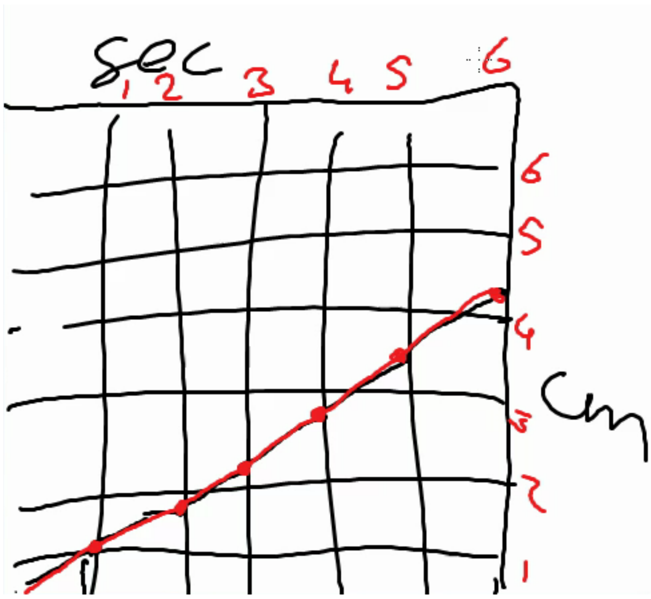


Figure 4.9: Graph emphasized and annotated by the teacher

Following the activity to create a minimal model of filling the cocktail glass, the teacher invited several pairs to present their model in front of the class. One of these models was a graph-like model (Figure 4.9). This involved an exchange in which the students clarified their graph, and the teacher labeled the axes with units and numbers and emphasized the graph by drawing it again in red. Then, after a short detour discussing other students' models, the teacher returned to the graph-like model and asked a student (Eric) if he could find the rising speed in the graph:

```
Eric    :   No
Teacher :   Why not?
Eric    :   Because it's slanted and it's slanted like that the whole time.
Teacher :   And that fits with this, with this, this glass?
Eric    :   No.
Teacher :   No, right? And how should it go then, you think? Eric, or Larry
Larry   :   I think it should go a bit bent
```

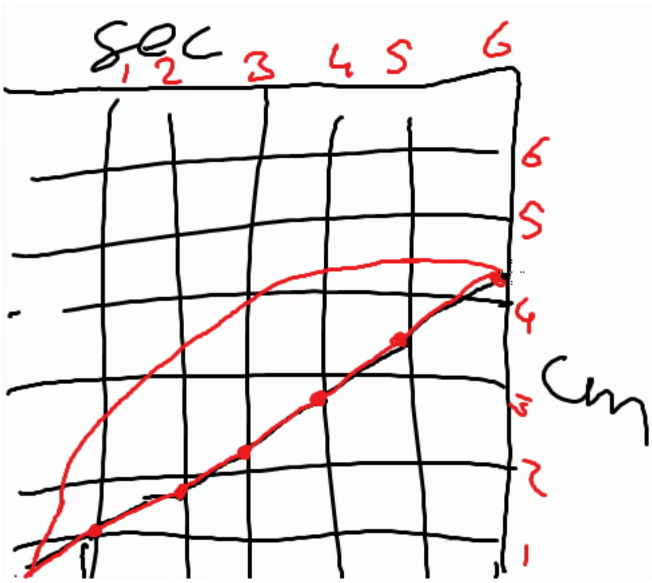


Figure 4.10: Larry draws a concave curve on top of the original straight line

While drawing a curve in red on top of the graph (see Figure 4.10), Larry explained that at a certain moment the graph almost does not rise any more. When the teacher asked if everyone understood what Larry was doing, they answered yes; Jessica explained:

```
Jessica :   Yes, because it is bent, you can have it go, yes, steeper,
            so you can specify that it goes slower all the time. Because
            in the end, it has to go to the right.
Teacher :   But, where, where, in what part of the graph it rises the
            fastest, for example?
Timothy :   In the beginning, at the bottom
```

```
Teacher :   Yes, in the beginning, here at the bottom. But why, how can you
            see that?
Timothy :   Well, because there, it goes up.
Teacher :   But here (teacher points to a part of the graph at two thirds of
            the length of the graph) it does also go up
Timothy :   Yes, but it goes a little less higher
Teacher :   Less higher?
Timothy :   Yes, less fast
```

end of vignette

Students are smooth thinkers, but fall back to discrete models We explored this unexpected difference between the two classrooms in detail in Chapter 3, where we concluded that although classroom conditions often gave rise to the invention and use of discrete representations, the underlying conception that the students reasoned from was continuous. Various facts substantiated our conjecture that the students came to the classroom with a continuous conception of speed. First, the students quickly realized that the speed was continuously decreasing because of the growing width of the cocktail glass's bowl, when they saw a cocktail glass fill up. Second, the students spoke about their realistic drawings or snapshots models as descriptions of a process. Third, the ease with which the students came up with snapshots suggested that students were familiar with the idea of using discrete snapshots to visualize change over time. Fourth, the students effortlessly accepted the curve, once it was introduced, which could be explained by assuming that the continuous model matched their existing continuous conception of the situation. Fifth, after having seen how a discrete graph approached a curve when decreasing the interval between points, the students never referred back either to the bar graph or the arrow graph in later lessons. Finally, the students in classroom C1 developed a continuous curve by themselves because they were dissatisfied with the discrete and linear graph.

Procedural understanding and quantification With regard to students' superficial notion of quantification, the teaching experiment of design experiment 3 further showed that the students were able to quantify speed with the help of the teacher. However, their quantitative understanding never seemed to exceed the ability to execute a procedure to compute speed. Apparently, the quantities, the units, and the measurements were not meaningful to them, and did not transcend the level of numbers in a calculation.

Finally, with respect to the flexibility of the students' understanding of instantaneous speed, the way the students reasoned in the context of cooling and heating was similar to that in the context of filling glassware. They were able to represent their understanding of a situation in a graph and could interpret a graph in light of the situation. They were able to point out when the speed was highest or lowest, and could use the tangent-line-tool to measure instantaneous speed.

We may briefly summarize what we learned:

- The students in C1 were able to invent a continuous graph by themselves, by using their understanding of the relationship between a glass' width and speed, and thus of the character of the covariation. Therefore, they are not chunky thinkers; they are continuous thinkers who have difficulty with graphing!
- Although starting discrete, the students in classroom C2 came to understand the continuous graph via shrinking the intervals of a bar chart.
- The students construed the tangent line as an indicator of the speed at a given point. (We see this as a tangent line; the students however were not yet familiar with that notion.)
- Students came to understand the shape of the graphs of a cooling process as being predicted by the tangent line that is depended on the temperature differences.
- Students did not have a sound basis for calculating speeds: they did not really understand average speed, and they had a limited understanding of the measurement units. They need more experience with quantifying constant speeds in a variety of ways (with different units), and relating those with the corresponding graphs.

4.3 The yield and trackability of our learning process

We may argue that the instructional sequence we tried in the last teaching experiment worked rather well. Although the students' weak understanding of measuring speed clearly was a limiting factor. On the other hand, the invention of the curve as a better representation of the process of filling a cocktail glass surpassed our expectations. If we adjust for these two aspects we may describe the prototypical instructional sequence emerging from the last teaching experiment as follows:

1. Students model the process of filling a cocktail glass.

2. Supported by a computer simulation, students develop a minimalist model of this process on paper. This explicates the underlying process: the wider the glass, the slower the water level rises.

3. Students work towards conventional graphs that are known to them: bar graphs and segmented-line graphs.

4. Students are supported in inventing the continuous and curved line graph as the best representation of the cocktail glass' filling process by a) critiquing the segmented graph and b) increasing the number of bars in the bar graph with a computer tool.

5. Students explore the highball glass, its linear graph, and its constant speed with the computer tool (that can show a simulation of the filling process, and has the option of showing the corresponding graph); by investigating highball glasses with various widths, and by varying the measurement units.

6. Students discuss the question: "When are the speeds in a cocktail glass and a highball glass the same?"

7. Students are introduced to and work with a virtual highball glass in the computer simulation.

8. Students compare the graphs of the cocktail glass and the highball glass; they figure out that a straight line that touched the cocktail glass' curve at the point where the widths of both glasses are the same is parallel to the highball glass' graph, and it signifies the speed in that point.

9. Students investigate the tangent line as an indicator of speeds in various points of a graph; both static, in a given point, and dynamic, moving along the graph.

10. Students generalize their understanding of instantaneous speed and graphs towards the context of heating and cooling of water.

The development of this prototypical instructional sequence was illustrated in the previous section in terms of our own learning process, which encompassed both the design decisions and the rationale for those decisions, and thus also described the emerging LIT. To avoid too much overlap, we will not reiterate this LIT here in detail, but summarize the most important theoretical findings. We formulate such a LIT elsewhere in detail in Chapter 5, however.

1. Key is the thesis that the students are spontaneously thinking in terms of instantaneous speed from the start. This conception allows the students to come up with "equal widths" as the answer to the question, "When is the speed in the cocktail glass and the highball glass the same?" This also fits with the students' quick understanding of the constantly decreasing speed in the cocktail glass, and with the intuitive appreciation of the curved continuous graph. It also enables the students to reinvent the continuous graph by arguing that the graph of filling a cocktail glass can not be linear because the rising speed is diminishing all the time. Thinking of speed as instantaneous speed, and realizing that the speeds are the same at the point were two glasses have the same width, enables them to use the virtual highball glass as a means of establishing the speed at a given point in a cocktail glass.

2. When asked to turn a highball glass into a measuring cup, the linearity of the relationship between water level height and volume is self-evident for the students. Having made such a measuring cup, they effortlessly express the imaginary process of the glass filling up with a linear graph. As the students have little understanding of average speed, we may assume that they think of constant speed in connection with a linear process. This understanding is powerful enough to help the students construe the tangent line as an indicator of the speed in a given point: by combining the cocktail glass' graph and the graph of the virtual highball glass. Unfortunately, their understanding of, and fluency with measures for speed hinders the opportunity to subsequently quantify instantaneous speed. However, they understand the tangent-line-tool well enough to link the difference between the actual and the final temperature to the tangent line in a cooling or heating process, and explain the cooling (respectively heating) graph in this manner.

These theoretical findings offer the rational underpinning the prototypical instructional sequence. They can be taken as conjectures on how students in other classrooms may reason. In this manner, they offer a framework of reference on the basis of which teachers may adapt the instructional sequence to their own needs and their own classroom. In addition, they offer support for instruction design, and for new design experiments and further theory development.

4.3.1 Empirically-grounded theory

In order to function in this manner the aforementioned theories have to be substantiated. Following Smaling (1990) we argued earlier that the theoretical findings could be substantiated by the virtual repeatability of one's research by other researchers. In relation to this, we have to bear in mind that design research differs from more classical forms of research in what it tries to achieve. A standard goal for classical educational research is to establish whether intervention A works better than intervention B. The goal of design research, however, is quite different. Here the goal is to generate a theory on *how* the intervention works. This distinction may be linked to two conceptions of causality, as described by Maxwell (2004): a regularity-oriented causality, and a process-oriented causality, respectively.

The former, which looks at the frequency with which an assumed cause is followed by a presumed effect is typical for quantitative research in education. In contrast, the kind of research we are discussing here employs a process-oriented perspective on causality, which 'deals with events and the processes that connect them; it is based on an analysis of the causal processes by which some events influence others.' (Maxwell, 2004, p. 5) It is based on the idea that causation is a complex process that is dependent on a combination of variables and circumstances. In principle, causal claims could be based on a single case. When aiming at LITs, a single case will not be just one student. Instead, the classroom as a whole will be the unit of analysis. As a result, to justify that there is a causal relation, narrative explanations can be used (Abell, 2004). Applied to educational settings, Maxwell (2004) argues that '[t]o develop adequate explanations of educational phenomena (...), we need to use methods that can investigate the involvement of particular contexts in the processes that generate these phenomena and outcomes.' (p. 7) We may argue that design research deploys two such methods: validating existing conjectures and generating new explanatory conjectures.

4.3.2 Validating and generating conjectures

In each design experiment, the conjectures about the learning process and the means of support are substantiated in the instructional activities and materials that are developed, tried out, adapted, and refined in multiple micro-design cycles during the teaching experiments phase of the design experiment. Conjectures that are confirmed by the students' actual learning process remain part of the LIT and are tried and refined again in the next design experiment. As a result, conjectures are confirmed or rejected in multiple different situations, offering a form of triangulation that adds to our understanding of students' learning processes in terms of these conjectures. Those observations enabled us to develop some theories about the mechanisms that were at play here.

In addition to this, new explanatory conjectures are generated through abductive reasoning, which is to 'rationalize certain surprising facts by the adoption of an explanatory hypothesis' (Fann, 1970), allowing us to focus on causal processes that we did not anticipate in the LIT. A surprising fact is an indication that our understanding of the students' learning processes did not match the actual students' learning processes; it suggests a gap in our theory. To fill that gap, we select data during the retrospective analysis that allows us to focus on what happened during the unexpected event to explore plausible hypotheses of why it happened. By detailing our learning process, we may show why and how we generated new explanatory conjectures.

Our learning process described in the previous sections shows a chain of reasoning throughout the subsequent design experiments focusing on the dichotomy between students' discrete and continuous reasoning and the lack of conceptual whole-class discussions about speed. In design experiment 1, students reasoned about the continuous changing non-linear situation of the cocktail glass from a discrete perspective. While exploring what happened, we also found that there was a lack of conceptual discussion about speed, adding to our difficulty explaining why students reasoned from a discrete perspective. We found a plausible hypothesis in Castillo-Garsow's (2012) distinction between chunky and smooth thinkers: and we conjectured that the students' discrete thinking was reinforced by a discrete learning environment. We decided to adapt the initial LIT in design experiment 2 by offering students a continuous learning environment, while using a modeling-based learning approach to graphing to provoke conceptual discussion.

Through the modeling-based learning approach we gained more insight in students' conceptions about instantaneous speed. However, our adaptation of the learning environment by focusing on exploring the process of filling the cocktail glass, proved less effective. During the first modeling activities, students created increasingly discrete models and, although one pair of students introduced a graph-like model that the other pair appropriated in a later activity, students had trouble to link the curve to their developing understanding of speed. This suggested that the students were chunky thinkers (Castillo-Garsow, 2012) who saw change as fundamentally discrete in nature and need support switching to a continuous perspective.

We adapted the LIT in design experiment 3 by introducing an activity where students would, guided by the teacher, rediscover the curve step by step while building on their own models. Whereas this anticipated learning trajectory was followed in one classroom, in the other, the students themselves constructed the curve. This made us reconsider what we knew about the students' use of the word "speed", and we realized that we implicitly assumed that the students were thinking of average speed when they talked about speed. This led us to reexamine our data on the students' conceptions of speed, which lead to the conclusion that they did not speak of, or reason with, average speed. Instead, our analysis showed that they reasoned from an informal conception of instantaneous speed. Thus their intuitive notions of speed were continuous, but they lacked the representational competency to model it with continuous models, and therefore resorted to discrete snapshots models (Chapter 3).

The above shows how surprising facts evoked abductive reasoning, which generated potential explanations which could be tested on the available data.

4.4 Discussion

Even though some of those findings reflect what is already known about what students can do and how they are taught in regular classrooms, we believe that the findings as a whole offer a new and exciting perspective on what and how primary-school students learn about speed. Their insight in what determines the rising speed in a glass, and their implicit conception of speed as an instantaneous speed that may change continuously, can be exploited by supporting various forms of graphing, in order to develop a deeper understanding of speed that may evolve into understanding speed as a rate.

As a caveat we may note that these claims are based on observations of how the students acted and reasoned during the design experiments. While our interpretations of those observations fed into either the next activity or the next experiment, which resulted in micro-design and macro-design cycles that had strong similarities with an empirical cycle of hypotheses testing. We can, however, only make claims about the students who participated in our experiments. Even then we have to take care to carefully ground our claims in the observational data. That is, we have to make sure that our conclusions are valid for the majority of the students and not just a few, which is not always easy as not all students speak up during classroom discussions. Here it is important to determine what the social norms in the classroom are. We have to know, whether the students feel obliged to speak up when they disagree or do not understand what is going on. We judged that this was predominantly the case in the classrooms where the experiments were carried out.

In addition to the micro-design cycles that immediately contribute to the learning process of the researchers, one usually also carries out a retrospective analysis after the experiment is concluded. Here Glaser and Strauss' (Glaser & Strauss, 1967) method of constant comparison is the designated method. We especially applied this method in the third teaching experiment. Basically this method boils down to (a) developing conjectures about what happened during the experiment, and test those against the available data, and (b) developing conjectures about why this happened, and test those conjectures against the available data.

Mark that the results of this procedure limit themselves to statements about the actual design experiments. Glaser and Strauss's (1967) method does not answer the question whether those results are generalizable. On the basis of those findings, however, a design experiment can be treated as a paradigm case. The goal then is to come to understand (the role of) the specific characteristics of the investigated learning ecology in order to develop theoretical tools that make it possible to come to grips with the same phenomenon in other learning ecologies. Instead of a scripted lesson plan, the LIT offers a theory of how the intervention works, which teachers and instructional designers can adjust and adapt. To be complete, we should add that design research may also aim at developing theories—or theoretical tools—on more encompassing issues that transcend the LIT, such as 'classroom social norms', or 'symbolizing'. In such cases, the learning ecology, which is created, serves as a condition for research on this more encompassing issue.

4.4.1 Educational design and design research

Finally, we want to discuss the relationship between educational design and design research we touched upon in the introduction. Design research uses educational design as a tool to realize a learning ecology that is conductive to exploring students' learning processes in detail. In particular, educational design comes to the fore during the teaching experiment phase of a design experiment. In this phase an educational design is created on the basis of a conjectured LIT: In multiple micro-design cycles, instructional activities are developed, tried in a classroom setting, evaluated, adapted, and refined. In this phase, although the LIT is leading, the emphasis is on providing meaningful instruction to the students. Ultimately, however, although design research does produce an instructional sequence, design research aims at developing a LIT that explains how that instructional sequence works in terms of students' learning processes.

The instructional sequence does not often evolve beyond a prototype in design research (Burkhardt & Schoenfeld, 2003), which implies that it cannot directly put into practice. From a design research perspective, the theory that is produced is more important, as it may inform researchers, educational design practitioners, and teachers about students' learning in a particular domain and means to support that learning (Gravemeijer & Cobb, 2013). However, from a practical point of view, the potential utility of the LIT is most apparent through the plausibility and support of the instructional sequence. This difference in aim, theory or practice, lies at the heart of the divide between the communities of design research and educational design (Schoenfeld, 2009). We think that design research practitioners can help bridging this gap by contributing to the codification of design practice called for in the educational design community (Schunn, 2008; Schoenfeld, 2009). In this respect we may point to the theory of realistic mathematics education (RME). RME is developed by generalizing over LITs (Treffers, 1987), and can be recast in terms of a set of instructional design heuristics (Gravemeijer, 2008).

To develop theory implies a commitment to the credibility of the LIT by justifying the claims made in it and a commitment to allow other researchers to assess the trustworthiness of the process leading up to those claims. The latter commitment can be satisfied by virtual replicability or trackability (Smaling, 1990), which means giving a detailed account of the design research process and the researchers' own learning process embedded in it. This will encourage design researchers to pay attention to practical issues of design. Hopefully, this will allow design researchers to create theories that are more practical applicable.

On the other hand, design research can be taken as a paradigm that may show educational designers the value of documenting design decisions and anticipated students' learning processes. Similar to design researchers, they may start describing their learning processes by making their design decisions explicit when preparing for an educational design project. While also explicating how these decisions build on existing theory as found in the literature. Furthermore, during the retrospective analysis, they may document how they evaluated what happened during the teaching experiments what they did to optimize the design, and how they thought about why the design worked the way it did. From this perspective, design research can be seen as an augmented form of educational design, which offers educational designers indications on how to handle the issue of documenting their practices and knowledge.

5

A proposed LIT for teaching instantaneous speed in grade five[*]

5.1 Introduction

In reaction to the advent of the information society, some researchers argue the need for a new and innovative science and technology (STEM) education in primary school (Léna, 2006; Gravemeijer & Eerde, 2009; van Keulen, 2009; Millar & Osborne, 1998). With the advent of ubiquitous networked computing devices connected to sensors, the importance of understanding dynamic phenomena in every-day life is growing. Key to a better understanding of dynamic phenomena is the concept of instantaneous rate of change. In the context of primary school we prefer to speak of "speed" because students of this age will not have developed the level of abstraction associated with the use of "rate of change" in the literature. Currently, instantaneous speed is not part of the primary school curriculum; conventionally, instantaneous rate of change is taught first in a calculus course in late high school or college. To meet

[*]This chapter is based on Beer, H. de, Gravemeijer, K., & Eijck, M. van. (submitted) A Proposed Local Instruction Theory for Teaching Instantaneous Speed in Grade Five.

the requirements of the information society, however, one may argue for teaching this topic much earlier. In response to this idea, we carried out a series of teaching experiments aimed at developing, trying out, and improving a local instruction theory (LIT) on instantaneous speed in grade five. The aim of this chapter is to propose a potentially viable LIT on this topic on the basis of those experiments.

After briefly discussing the literature on primary students' conceptions of speed and teaching calculus-like topics early in the mathematics curriculum, we elaborate on the method of design research used in our research to develop the proposed LIT for teaching instantaneous speed in grade five. Before formulating that proposed LIT, however, we discuss the results of our design research project in terms of patterns emerging from the data regarding students' key learning moments. Finally, we discuss the proposed LIT by placing it in the literature and pertaining to its utility.

5.2 Theoretical background

In both the literature on primary school students' conceptions of speed and the primary school curriculum, speed is mainly treated as an average speed and interpreted as a ratio of distance and time. Students have problems with relating distance and time to speed (Groves & Doig, 2003), and 5th grade students have trouble developing an understanding of speed as a rate (Thompson, 1994b). Stroup (2002) attributes these problems to conventional approaches to teaching speed that favor ratio-based understandings of speed over students' intuitive understandings. Instead, he argues that developing students' qualitative conceptions of speed is a worthwhile enterprise in itself. This "qualitative calculus" approach further deviates from conventional approaches by starting with exploring non-linear situations of change (Stroup, 2002). Young students appear to have an intuitive understanding of non-linear situations even before those are taught explicitly (Ebersbach & Wilkening, 2007; Confrey & Smith, 1994). In most studies on primary school students' conceptions of speed, however, instantaneous characteristics of speed have not been explored.

Because instantaneous speed is usually taught first in a calculus course, we reviewed the literature on teaching calculus-like topics early on in the mathematics curriculum as well. This literature is part of a longer tradition of studying calculus reform (Tall et al., 2008). However, most of these reform initiatives can be characterized as 'largely a retention of traditional calculus ideas now supported by dynamic graphics for illustration and symbolic manipulation for computation.' (Tall, 2010, p. 3) As a result, most of these initiatives do not offer a solution to one of the main barriers for learning calculus, the limit concept (Tall, 1993). Meyer and Land (2003) characterize limit as a threshold concept. Gaining understanding of a threshold concept is of fundamental importance to being able to deepen understanding of advanced concepts that build on that understanding (Perkins, 2006). However, Perkins (2006) argues, threshold concepts could 'get ritualized by students or indeed teachers, and in general present a hurdle' (p. 44). Most calculus reform initiatives focused either on improving traditional calculus courses or changing the mathematics curriculum in service of teaching and learning calculus in the formal mathematical sense. Other researchers, however, went beyond the educational implications outlined by traditional calculus and focused on learning of calculus-like concepts already in primary school (Nemirovsky, 1993; Thompson, 1994b; Boyd & Rubin, 1996; Nemirovsky et al., 1998; Noble et al., 2001; Stroup, 2002; Ebersbach & Wilkening, 2007; van Galen & Gravemeijer, 2010).

In this literature, two characteristics seem to be shared: computer simulations and Cartesian graphs. Research suggest that young students are able to explore speed mathematically using computer simulations, which are a natural fit to explore dynamic phenomena. Computer technology enables inquiry-based learning approaches because it allows students to explore more authentic, realistic, and complex problems (Chang, 2012; Ainley et al., 2001). However, because computer simulations can also function as black boxes, Gravemeijer et al. (2000) argue that computer simulations only induce adequate learning processes when embedded in a suitable instructional sequence. If that instructional sequence and computer simulation offers support for graphing as well, students will be able to express their understanding of dynamic situations through graphs (Phillips, 1997; Ainley et al., 2000; van den Berg et al., 2009), even though graphing is a marginal topic in primary school and older students encounter all kinds of problems (Leinhardt et al., 1990). More specifically, Roth and McGinn (1997) propose social-cultural approaches to graphing to support students in becoming practitioners of graphing; alternatively, diSessa (1991) proposes to support students in inventing graphical representations rather than introducing them to ready-made representations.

In our design research project, we developed such a suitable instructional sequence supporting students to use computer simulations and graphs in service of exploring the concept of instantaneous speed.

5.3 Method

5.3.1 The design of the initial LIT

The theory of realistic mathematics education (RME) (Treffers, 1987; Gravemeijer, 1999) is used in designing the LIT. Central to RME theory is the adagio that students should experience mathematics as the activity of doing mathematics, and should be supported in reinventing the mathematics they are expected to learn. In relation to this, three instructional design heuristics are formulated: guided reinvention, didactical phenomenology, and emergent modeling (Gravemeijer & Doorman, 1999). According to the guided-reinvention heuristic, a route has to be mapped out that allows the students to (re)invent the intended mathematics by themselves. Here, the designer can take both the history of mathematics and students' informal solution procedures as sources of inspiration. In the case of speed, history shows us that intuitive notions of instantaneous velocity preceded the conception of instantaneous speed as the limit of average speeds on increasingly smaller intervals. This suggests to try to build on this intuitive notion of instantaneous velocity. The choice of guided reinvention as our point of departure is intertwined with the way we frame our goals. Our primary goal is for the students to develop a framework of mathematical relations, which involve co-variance, tangent lines, rise-over-run, and eventually, speed as a variable.

The didactical phenomenology heuristic advises one to look for the applications of the mathematical concepts of tools under consideration, and analyze the relation between the former and the latter from a didactical perspective. In this case, the phenomenon of the speed of rising water in the context of filling glassware comes to the fore as a suitable point of impact for the reinvention process. The didactical phenomenology also advises a phenomenological exploration to ensure a broad conceptual base. However, the small number of lessons did not allow us to do justice to this idea.

The emergent-modeling design heuristic asks for models that can have their starting point in a *model of* the students' own informal mathematical activity, and can develop into a *model for* more formal mathematical reasoning. Key in this transition is that the students develop a network of mathematical relations, while acting with the model. In this manner, the model may begin to derive its meaning from the mathematical relations involved, and may start to function as a model for more formal mathematics reasoning. In our case, the model may loosely be defined as "visual representations of the filling process". These representations may first come to the fore as models of water heights at consecutive time points, and gradually evolve into a model for reasoning about the rising speed.

Applying those design heuristics resulted in a preliminary LIT, which was elaborated and improved in a series of three design experiments. This yielded a LIT based on the following ideas. To circumvent the highly complex idea of the limit, we try to capitalize on the students' intuitive notions of instantaneous speed in the context of filling glassware. The context of filling glassware is chosen because students of this age understand that the rising speed at a given point is determined by the width of the glass. This implies that they have a conceptual basis for thinking through how a filling process for a given glass will evolve. Thus they have a conceptual basis for thinking about how the dependent and the independent variable co-vary. We assume that the students dispose of an implicit notion of instantaneous speed.

The instructional sequence aims at supporting the students in expanding their informal notion of instantaneous speed. Following the emergent modeling design heuristic, we turn to graphing filling processes as a catalyst for framing instantaneous speed as a topic for discussion. The curvature of the graph of filling a cocktail glass, for instance, signifies the instantaneous character of the rising speed. The graphs may first come to the fore as models of water level heights at consecutive time points. With support of the teacher, a computer simulation tool (which offers various dynamic representations that fit with the steps in the learning process), and the tasks, the graphs gradually evolve into a model for reasoning about the rising speed. Here the students' understanding of the relation between the glass's width and the rising speed is exploited by asking the students when the rising speed in a cocktail glass is the same as the rising speed in a (cylindrical) highball glass. Once this relation is established, the students can link the instantaneous speed at a given height in the cocktail glass with the constant speed of a corresponding highball glass; and by extension, they can link the linear graph of a highball glass to a point of the graph of a cocktail glass. Then, with the help of the computer simulation tool, the students can calculate the speed that corresponds with the tangent line at various points and move the point with the tangent line along the graph to get a sense of speed as a variable.

5.3.2 Design research

Because design research is well-suited to explore teaching of topics in earlier grade levels than they are usually taught (Kelly, 2013), we use this research methodology to explore how to teach instantaneous speed in5th grade.

As a literature review did not offer us enough support to formulate an initial LIT,

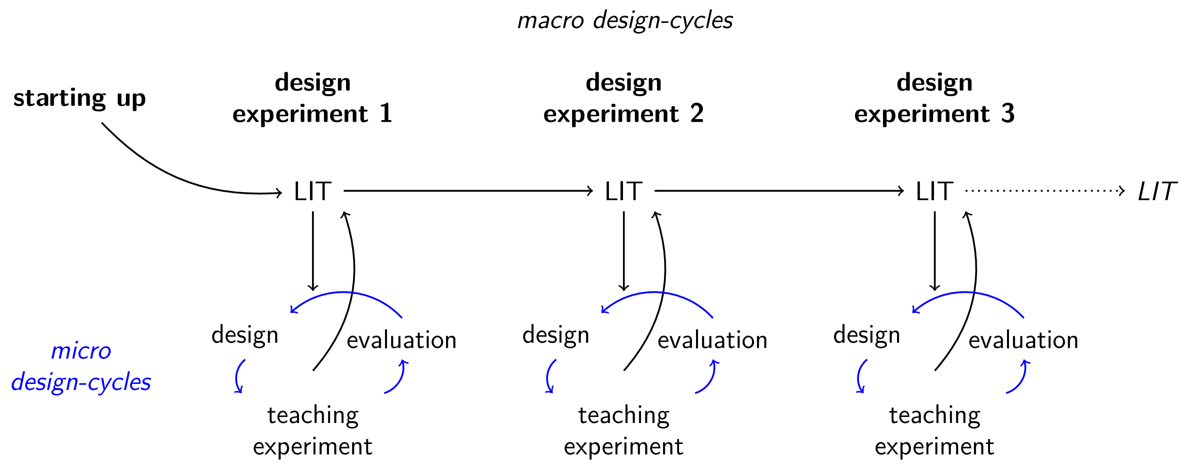


Figure 5.1: Schematic overview of our design research project aiming at developing a LIT on teaching instantaneous speed in 5th grade. (Taken from de Beer, Gravemeijer, and van Eijck (in preparation-a))

we first performed a series of one-on-one teaching experiments (Steffe & Thompson, 2000), in order to explore 5th graders' prior understanding of speed in situations with two co-varying quantities (Chapter 2). This allowed us to formulate an initial LIT, which we elaborated, adapted, and refined in three macro design-cycles called design experiments (Cobb, 2003; Gravemeijer & Cobb, 2013) (see Figure 5.1). Each design experiment starts by designing an instructional sequence that instantiates the ideas envisioned in the LIT. Next, that instructional sequence is tried in a teaching experiment that consists of a sequence of micro design-cycles of (re)developing, trying out, evaluating, and adapting instructional activities and materials (Figure 5.1, blue cycles). A design experiment is completed by a two-step retrospective analysis that is modeled after Glaser and Strauss's (1967) comparative method, in particular, the elaboration of Cobb and Whitenack (1996) on this method was used: After formulating conjectures about *What happened?* during the teaching experiments and testing these conjectures against the data collected, a second round of analysis is carried out by formulating conjectures about *Why did it happen?*. These conjectures are also tested against the data collection. The results of the retrospective analysis are used to adapt and improve the LIT, which is input for the next design experiment. For more information about the retrospective analysis phase of a design experiment, we refer to Chapter 3, wherein we elaborate on the retrospective analysis of the third design experiment, showing how new explanatory conjectures are generated to improve the LIT.

In Chapter 4, while pertaining to the trackability (Smaling, 1990) of our research, we give a detailed description of the development of both a prototypical instructional sequence on instantaneous speed and the accompanying LIT during the three subsequent design experiments. In this chapter, we shift our focus away from the process of doing design research to presenting the findings of that process: the proposed LIT (Figure 5.1, rightmost side).

We will formulate the proposed LIT based on the argumentative grammar for design experiments presented by Cobb et al. (in press). Although their argumentative grammar treats the justification of the theoretical findings of a *single* design experiment, we think it can serve as a guide to justify the findings of the whole design research as well given the iterative nature of design research. After all, the latest design experiment builds directly on the findings of previous design experiments. However, instead of treating the students' learning processes in the last design experiment alone, we treat the students' learning processes that comes to the fore throughout all design experiments, with an emphasis on the last design experiment. To do so, we first identify patterns in students' learning processes that emerged from the data from the whole design research project.

According to Cobb et al. (in press), the justification of the theoretical findings of a design experiment follows from a) showing that the students' learning process is due to their participation in the design experiment, b) describing that learning process, and c) enumerating the necessary means of support for that learning process to occur. As Cobb et al. (in press) observe, given the nature of design research to explore innovative learning ecologies, the first aspect of the argumentative grammar is often trivial. In this case it concerns: the concept of instantaneous speed, which is not part of the primary curriculum and quantification of instantaneous speed is conventionally the domain of calculus; without their participation in our various teaching experiments, the students would not have developed a more mathematical notion of instantaneous speed. The other two aspects of the argumentative grammar closely relate to the conception of a LIT. The difference between the two is that the LIT describes the theory, whereas the argumentative grammar asks for documentation. We address those differences by grounding the LIT in the patterns that emerged in the data.

During our design research process, we formulated various conjectures about the students' learning process in service of developing and improving the LIT on teaching instantaneous speed in 5th grade. Initially, these conjectures were formulated based on the results of the starting-up phase and functioned as the starting point for the first design experiment. In each design experiment the conjectures in the LIT were put to the test in multiple teaching experiments while data was collected from multiple sources. The teaching experiments were recorded on video, whole-class discussions on those videos were transcribed, student products were collected, in small-scale teaching experiments the students' computer sessions were captured, all meetings with the teacher were recorded on audio, and observations were made during and after the lessons. Although all data sources contributed to our understanding of students' learning processes, the transcriptions of whole-class discussions and student products in particular formed the basis for the retrospective analysis that followed. The conjectures that validated our understanding of what happened and why it did happen were carried over to the initial LIT of the next design experiment. The conjectures that were refuted were taken as indications of a mismatch between our understanding of the students' learning process and their actual learning process. To improve the LIT, we generated new explanatory conjectures through a process of abductive reasoning (Chapter 3) and added them to the initial LIT of the next design experiment.

During this process of validating and generating conjectures about the students' learning processes some patterns emerged. The justification of these key learning moments stems from a triangulation on two levels. Besides the use of multiple data sources in the generation and validation of the conjectures during a single design experiment, the validation of these key learning moments also recurred over multiple design experiments (Chapter 4), strengthening the empirical basis of the results of design research.

5.3.3 Teaching experiments and data collection

The proposed LIT aims to be a potential point of departure for other researchers, educational designers, and teachers who are interested in teaching instantaneous speed early on in the mathematics curriculum, in particular grade five. The value of the LIT to these users is in its potential usability, which depends on the extent that the LIT is relevant to *their* situation. This criterion of transferability (Smaling, 2003) is only possible when potential users of the LIT understand the particulars of the design experiments well enough to assess the similarities between the context from which the LIT originated and their own situation (Smaling, 2003). A criterion that is often denoted as "ecological validity" (Gravemeijer & Cobb, 2013). Typically, a thick description is provided to facilitate that understanding (Gravemeijer & Cobb, 2013). This need for a thick description fits nicely with the requirement to identify the patterns in students' learning processes.

Before we enumerate and discuss these patterns in the next section, we want to make plausible that, despite the unique situation of the teaching experiments, the learning environment of the teaching experiments were not a-typical for a 5^{th} grade classroom by briefly discussing the teaching experiments. At the same time, we want to contribute to the ecological validity by enabling teachers and instructional designers to judge how the conditions of the teaching experiments differ from their classrooms.

We performed various teaching experiments in both the starting-up phase and in each of the three design experiments. We did a series of one-on-one teaching experiments in the starting-up phase and a small-scale design experiment 2 with, respectively, 9 and 4 average to above-average performing 5^{th} graders from two different primary schools. The first author acted as a teacher. These small-scale teaching experiments can be characterized as try-outs for the classroom teaching experiments in subsequent design experiments. In design experiments 1 and 3 we performed classroom teaching experiments with a gifted mixed $5^{th}/6^{th}$ grade classroom (25 students) taught by an experienced teacher in design experiment 1 and two gifted mixed 4^{th}-6^{th} grade classrooms (both 24 students) taught by the same novice teacher in design experiment 3. The teacher's role was executive in nature, but we discussed the lessons and the underlying rationale with them beforehand as well as afterwards to evaluate the lessons. At both schools, the gifted program was set up to pay attention to the students' special needs: to work on their social-emotional development, to learn meta-cognitive skills, and to improve their overall happiness about going to school. Nevertheless, the classes behaved like any other classroom; some students were better in mathematics than others, some students were more motivated than others.

5.4 Results

As indicated above, a retrospective analysis was carried out in each design experiment. This retrospective analysis consisted of two steps: 1. formulating conjectures about *What happened* and testing these conjectures against the data set, and 2. formulating conjectures about *Why it happened*, which were also tested against the data. In itself, the retrospective analysis of the last design experiment could function as the sole basis for the LIT that is proposed. However, in a secondary analysis, we went through the conjectures that were confirmed in one or more design experiments, looking for a pattern. Here we used the emerging LIT (see Chapter 4) as a lens to identify conjectures that signified "key learning moments".

Through analyzing the collected data of all teaching experiments— the videos of the whole-class discussions and the transcripts thereof enriched with the students' products formed the main data source (next to the protocols of the one-on-one teaching experiments in the starting-up phase); the other data mainly functioned as

	key learning moments										
	a	*b*	*c*	*d*	*e*	*f*	*g*	*h*	*i*	*j*	*k*
starting-up phase	×		×	×		×					
design experiment 1	×	×	×	×	×	×				×	
design experiment 2	×	×	×	×	×	×	×			×	
design experiment 3, C1	×	×	×	×	×	×	×	×	×	×	×
design experiment 3, C2	×	×	×	×	×	×	×		×	×	×

Table 5.1: (Re)occurrence of key learning moments during the subsequent teaching experiments in the design research project. The letters *a–k* refer to the key learning moments discussed in Section 5.4.2 (starting at p. 106)

a basis for triangulation—, patterns emerged with respect to students' key learning moments given students' instructional starting points. In Table 5.1 the (re)occurrences of these key learning moments throughout the teaching experiments are tabulated. The table shows how the empirical basis of the key learning moments is strengthened by the iterative and cumulative characteristic of design research, which increases the credibility of the results.

A limitation of course is that not all students contributed to the whole-class discussions. However, it appeared that in each design experiment there was a classroom culture in which the students felt safe to actively participate, freely express themselves, and ask questions. This suggests that many students who did not feel the need to participate in the discussion at least passively understood what was being discussed. Given the exploratory character of the study, we consider this a sufficient basis for construing a credible proposal for a LIT.

Before discussing these key learning moments in detail, we first describe the instructional starting points. We note that we have reported on some of these results before, but from a different perspective than we do now (Chapter 4), or to focus on just a small part of the whole design research (Chapter 3).

5.4.1 Instructional starting points

The patterns concerning the instructional starting points regarding speed and graphs was the same in each teaching experiment: the students had limited graphing experience, they had trouble computing speeds, and they were thinking in terms of instantaneous speed from the start. We have no indication that the situation will be significant different in any (Dutch) 5$^{\text{th}}$ grade classrooms. For example, Lobato, Walters, Hohensee, Gruver, and Diamond (2015) report similar behavior of students computing and using speeds in two USA schools as we observed during our teaching experiments.

Most graphs the students had encountered at school, which were not that many to begin with, were bar graphs, straight lines, or line graphs consisting of straight line segments to describe some more complex situation. The students did not have trouble drawing and interpreting a graph of the linear situation of filling the highball glass, which is a straight line. However, because the students were mainly acquainted with segmented-linear graphs as a means for describing co-variation of two quantities, when they were asked for the first time to draw a graph of what happens when a non-linear glass fills up, they either drew a straight line or some graph comprised of straight line segments.

Furthermore the students did not have a sound basis for calculating speeds: they did not really understand average speed, and they had limited understanding of the measurement units involved. Although the students did know the "divide distance by time" procedure to compute a speed in the motion context—it is part of the primary school curriculum, after all—, actually applying that procedure in the context of filling glassware appeared more difficult than we anticipated. With initial support of the teacher, the students were able to compute speeds this way. Even so, their command of units and quantities seemed lacking; consistently using appropriate units did require continuous teacher guidance. The following episode from design experiment 2 is a good example of students' struggle with computing speeds:

```
Teacher  : Okay. Could you determine the speed? How fast does
           the water rise? And what unit do we use, actually?
Maria    : You have to do something like distance divided by time,
           right?
Teacher  : Well, do you have a distance, do you have a time?
Maria    : I don't know. O, the time is 12 seconds
Teacher  : And the distance?
Maria    : 10
Teacher  : 10. And we don't really call it distance, because we don't
           talk about someone walking or bicycling, but we're talking
           about water that is rising. How fast does the water rise in the
           highball glass?
Maria    : 12 divided by 10
Roel     : 1.2
Teacher  : 10 divided by 12, apparently, that's 0.833 and so on.
Maria    : 0.813333333
Teacher  : We've got two different speeds. One is 0.8 cm/sec and the
           other is 1.2 sec/cm. Which one do you think is best?
Maria    : The bottom one (1.2 sec/cm)
Teacher  : Why do you want that one?
Patrica  : I don't know. That one is the largest, maybe?
```

```
Maria    : It looks easier, I think. 12 divided by 10, that's easier
```

To Maria, computing 1.2 cm/sec was preferred because dividing 12 by 10 is easier than the other way around (and therefore, apparently, more likely to be the "correct" one to compute). The students did not seem to care about the meaning of the division, nor the units involved. After this episode a lengthy discussion ensued about what unit to use for the speed of rising, if one at all. Although they decided on cm/sec, it is doubtful that this unit, or the use of any unit, was meaningful to the students. For example, they did not seem to realize that this ratio could also be represented in different units, such as meters per second. Throughout the design experiment, they had to be reminded to use units. The students in the other design experiments exhibited similar behavior.

Finally, a thorough retrospective analysis of all teaching experiments showed that the students were spontaneously thinking in terms of instantaneous speed from the start (Chapter 4). It showed that throughout the design experiments, the students appeared to understand filling the cocktail glass as a process with a continuous varying speed. Average speed did not seem to play a significant role, even though they were explicitly taught about the latter. Unsurprisingly, when first asked to determine the speed in the cocktail glass, they resorted to computing the average speed. However, they also understood that this speed was not an adequate measure of the continuously changing speed in the cocktail glass.

5.4.2 Students' key learning moments

The following patterns emerged with respect to the students' key learning moments (see Table 5.1):

a. *The students found the linearity of filling a highball glass self-evident.*

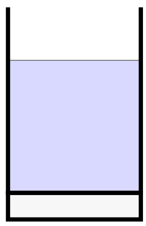


Figure 5.2: A highball glass

From the first teaching experiments in the starting-up phase, and in each subsequent design experiment, no student seemed to have trouble reasoning about speed in the linear situation of filling the highball glass (see Figure 5.2). They understood that the water level rises with a constant speed. They expressed that understanding, verbally, gesturally, and even graphically, indicating the uniformity with which the water level rises.

b. *The students had a very weak understanding of the corresponding measures of constant speed.*

From design experiment 1 onward, it appeared that the students had a shaky understanding of how to reason quantitatively about the speed in the highball glass. In each design experiment, after computing one speed together with the teacher, most students were able to compute speeds on their own. It was not uncommon for many students to treat the computed speed as a ratio. For example, they would write a computed speed down as "1 sec = 2 cm". Furthermore, it seemed that the students' skill in computing speed remained shaky and had to be re-activated in each subsequent lesson where speeds were computed.

c. *The students easily broke through the linearity illusion when they saw the cocktail glass fill up.*

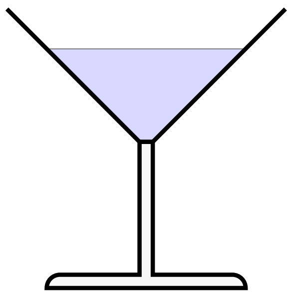


Figure 5.3: A cocktail glass

When the students were asked to imagine filling the cocktail glass (see Figure 5.3) and to draw a graph or make a model of that situation, most students created some linear model. These linear models were not so much an expression of their understanding of the situation as an expression of their implicit expectation of linearity due to the prevalence of the linear prototype in primary school. However, there were a small number of students in each design experiment that understood the situation to be non-linear from the start. While seeing the cocktail glass fill up, either an actual glass or in a computer simulation, all students seemed to easily break through this so-called "linearity illusion" (de Bock et al., 2002). After that, non-linearity became a salient characteristic for the students.

d. *Once they had been contemplating on the rising speed in a cocktail glass, the students realized the relationship between a glass' width and speed.*

In each design experiment and the starting-up phase, once the students started focusing on the non-linear nature of the situation of filling the cocktail glass, they realized that it would take a much larger volume of water to make the water level rise one millimeter at the top of the cocktail glass than at the bottom. Given a constant influx of water, they realized that the wider the cocktail glass, the slower the water level would rise. From the transcripts of the classroom discussions it appeared that this relationship between a glass' width and speed of rising was understood by most if not all students in each of the classrooms that participated in our design research project.

e. *The students needed little effort to come up with "equal widths", as the answer to the question, "When is the speed in the cocktail glass and the highball glass the same?"*

In each design experiment, when asked to come up with a strategy to determine the speed in the cocktail glass, the first suggestion that was proposed was to compute the average speed. From the discussions that followed, it seemed the students also understood that this was a bad measure for the changing speed in the cocktail glass. Computing the average speed on smaller intervals, as some students suggested in design experiments one and three, proved similarly problematic. No student was able to come up with a successful way to determine the instantaneous speed in the cocktail glass on their own.

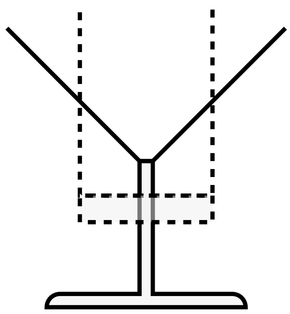


Figure 5.4: Imaginary highball glass as measure for instantaneous speed in the cocktail glass

However, in each case, once the teacher asked, "When is the speed in the cocktail glass and the highball glass the same?", some students immediately realized that when the highball glass and the cocktail glass have exactly the same width, they have the same speed. Soon, this notion seemed to become taken as shared. After having determined the speed at a given point with the support of the picture combining a cocktail glass and an imaginary highball glass (Figure 5.4), some students realized that this strategy could be generalized. The speed at any point could be determined by finding a highball glass that has the same width as the cocktail glass at that point and compute the highball glass' speed. These students invented the *imaginary highball glass* as a measure for instantaneous speed in the cocktail glass.

The students' understanding of how to utilize the highball glass this way was particularly convincing in design experiment 3 where the students had explored filling the highball glass in the previous lesson using a computer simulation with an extensible highball glass. Students could re-size this glass, allowing them to explore the relationship between a highball glass' size and its speed.

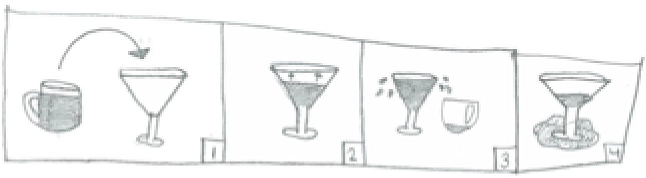
f. *The students were mainly acquainted with segmented-linear graphs as a means for describing co-variation, but were inclined to accept a continuous curve as a better representation for the way the water rises in a cocktail glass, once they had seen the cocktail glass fill up.*

Once the curve was introduced, most students accepted it as a better fit to describe what happens when the cocktail glass fills up than a straight line or a straight-lined segmented graph. More so, it appeared that most students realized the relationship between the steepness of the graph and the speed and the glass' width: the smaller the glass, the higher the speed, the steeper the curve. After seeing the computer drawn graph of the cocktail glass, most students were able to sketch or draw a reasonable accurate curve for other glasses as well. For example, in design experiment 3, when the students were asked to graph filling a cognac glass, after having seen the cocktail glass fill up, in general students drew a curve (11 of 12 graphs in classroom 1; 8 of 10 in classroom 2).

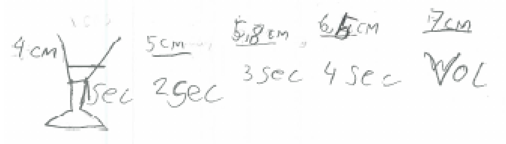
g. *When asked to model the rising speed in a cocktail glass, the students initially depicted rising water heights primarily with realistic snapshot models—which, with some help, they could connect with more conventional graphs.*

In design experiments two and three, we started with an iterative modeling process by asking the students to draw a model of filling the cocktail glass, and

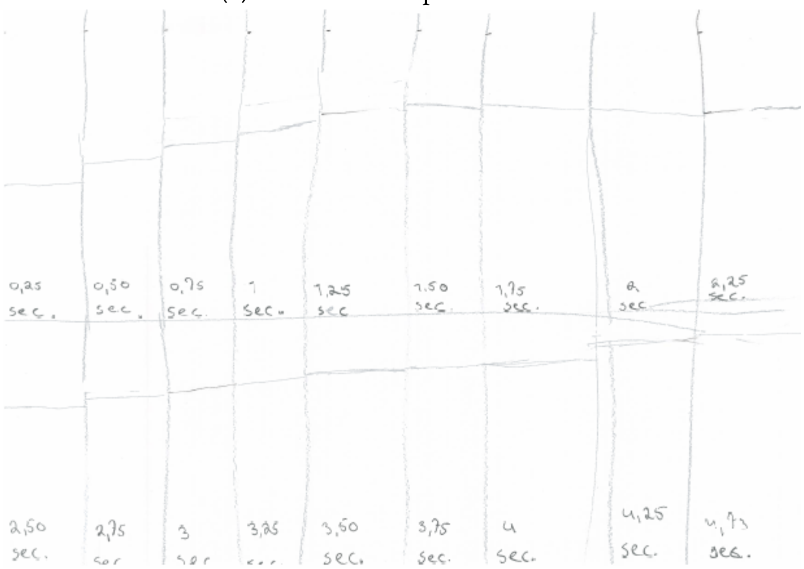
(a) Realistic depiction



(b) Snapshots model with realistic elements



(c) Table-like snapshots model



(d) Graph-like snapshots model

Figure 5.5: Design experiment 3. Models created during first modeling activity

asking them several times to try to come up with a better model. Initially, most drew quite realistic depictions of the situation while some made a snapshots model depicting subsequent moments (Figure 5.5-a,b). These more abstract snapshots models often included realistic aspects of the situation as well, such as a tap, droplets, or overflowing. After the students observed the cocktail glass fill up, almost all started paying attention to the non-linear nature of the situation. More and more students made snapshots models to express their understanding of that non-linear behavior. In the last modeling activity, for example, all students made snapshots models, some of which were table-like (Figure 5.5-c). In each classroom there were also one or two graph-like models (Figure 5.5-d) that looked like the straight line-segmented graphs the students were familiar with.

Once the graph was introduced, either by the students or by the teacher, the students seemed to see through it the snapshots model: with each points signifying a snapshot. These students were able to apply their limited experience with making and interpreting graphs, to interpret the cocktail glass' graph and to draw or sketch a graph of other glasses, such as a wine glass.

h. *The students were able to invent a continuous curve as a better representation for the process of the way the water rises in a cocktail glass, when reconsidering, and improving upon, a segmented graph.*

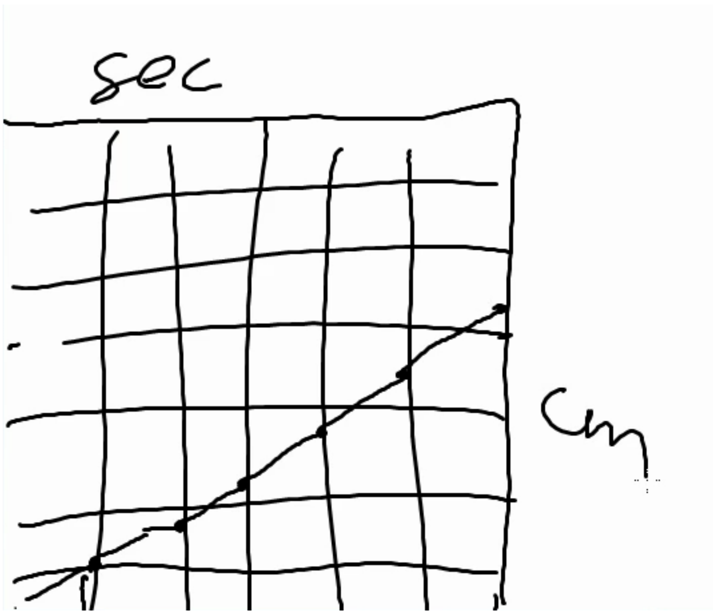


Figure 5.6: Graph-like model being discussed in class

In design experiment 3, classroom 1, the discussion of the models the students made in the last modeling activity revolved around a graph-like model. The students who made the model drew it on the interactive whiteboard in black (see Figure 5.6): a segmented straight line. The teacher re-drew the line in red to emphasize it. When he asked the students if they could find the speed in this graph, another student argued that a straight line did not fit his understanding of how the cocktail glass fills up:

```
Teacher: (...) If you look at this graph, right, what happens with the
         steep, the speed of rising? Can you read it here, read it back [from
         the graph]?

         Have a good look. What do you think, Eric?

Eric:    No
Teacher: Why not?
Eric:    Because it's slanted and it's slanted like that the whole time.
teacher: And that fits with this, with this, this [cocktail] glass?
Eric:    No.
teacher: No, right? And how should it go then, you think? Eric, or Larry
Larry:   I think it should go a bit bent
```

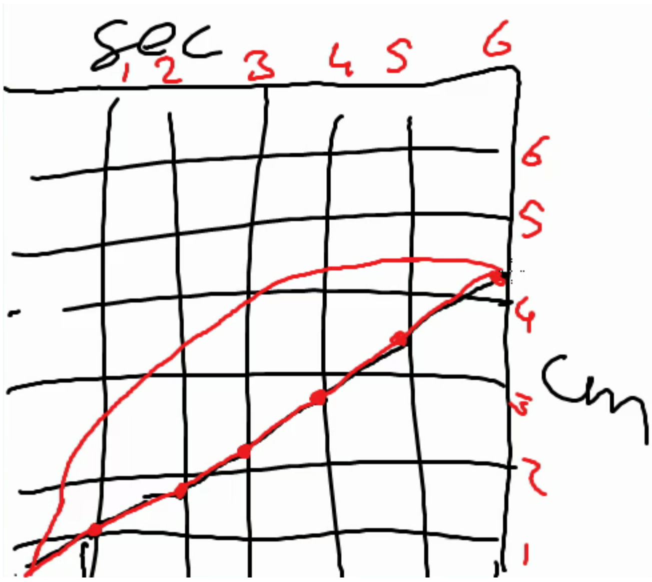


Figure 5.7: Larry drew, in red, a curve on top of the original straight line

Subsequently, the teacher invited Larry to draw his idea on top of the original graph (see Figure 5.7, red curve). Larry explained that at a certain moment the graph would almost not rise any more. When the teacher asked if everyone understood what Larry was doing, the class answered yes. The teacher invited Jessica to explain:

```
Jessica: Yes, because it is bent, you can have it go, yes, steeper,
         so you can specify that it goes slower all the time. Because
         in the end, it has to go to the right.
```

The above episode shows that these students themselves were able to invent the curve, even though most if not all graphs they would have encountered in school were either straight lines or straight-line segmented graphs. This suggests that, when suitably supported, other students could do so as well.

i. *The students could also develop the continuous curve via a guided process of shrinking the intervals of a bar chart.*

In design experiment 3, classroom 2, the teacher guided the students to discover the curve in a step-wise process. First, he introduced the bar graph of filling the cocktail glass, which the students connected to their own snapshots models: each bar represented a snapshot. Then, he focused students' attention to what happens in between two bars by drawing arrows in between each two subsequent bars. Although this had not the desired effect: the students kept on focusing on individual points. However, when the curve was finally introduced, the students accepted it as a better fitting model than either their own snapshots model or the bar chart.

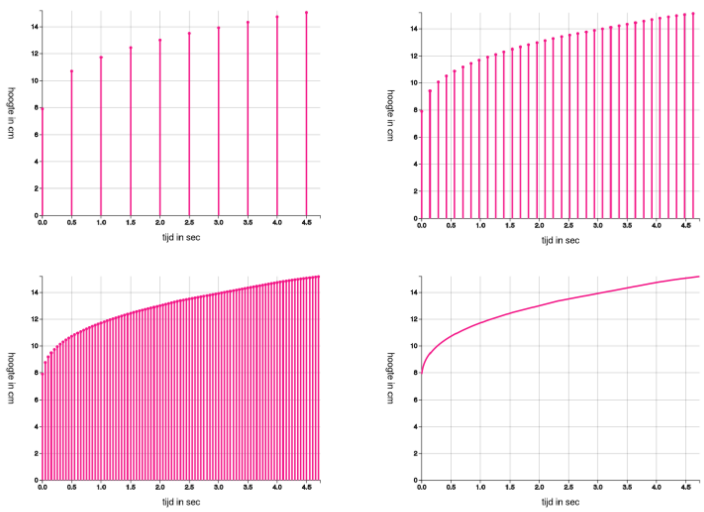


Figure 5.8: Interactively changing the interval between the bars in the bar chart

In classroom 1, however, after the students already had invented the curve, the teacher led showed them the bar chart as well. Different from the other classroom, he now increased the number of bars by shrinking the interval between the bars in the computer simulation. This was a powerful image for the students, who offered that it made the graph more precise and more clear. This led us to conjecture that by interactively shrinking the intervals between bars in the bar chart (see Figure 5.8), the students are supported seeing through the bar chart the pattern of change depicted by the curve as well.

j. *The students were able to construe the tangent line as an indicator of the speed in a given point by combining the cocktail glass' graph and the graph of a highball glass.*

   In each of the three design experiments, the students were introduced to the tangent line to a curve as a measure for instantaneous speed in a different way:

   1. In the first design experiment, a direct link was established between the highball glass as measure the students already knew and the tangent line. The tangent line was introduced as the imaginary highball glass' graph, connecting determining speed at a point in the glasses to determining speed at a point in the graphs (see Figure 9). That direct link was then severed and the glasses removed, leaving only the cocktail glass' curve and the tangent line.
   2. In the second design experiment, the students discovered the tangent line while discussing when and why the graphs of the cocktail glass and highball glass did have the same speed. Earlier in the discussion, some students discovered that drawing a straight line to a point on the curve does not result in a line with the same speed as the curve in that point (Figure 5.9, black wobbly line from the origin to (6s, 12.5cm)): that line does not have the same steepness as the curve in that point. Subsequently, the teacher reversed the problem and asked the students if they could indicate where on the curve the speed was the same as on the (brown, lower) straight line (Figure 5.10).

      The following episode followed:

```
Teacher: I ask you, when rises the water with the same speed in the
         cocktail glass as in this brown line? What would
         you pick, or select, or
Roel:    (Goes to the board) I'd pick about here and there (he points
         to a point around 8s on the brown line and a point around
         1.5s on the curve)
Teacher: All right, why there?
Roel:    Well, here (points to the brown line) it doesn't matter
         much, but about here (draws with his hand a tangent line to
```

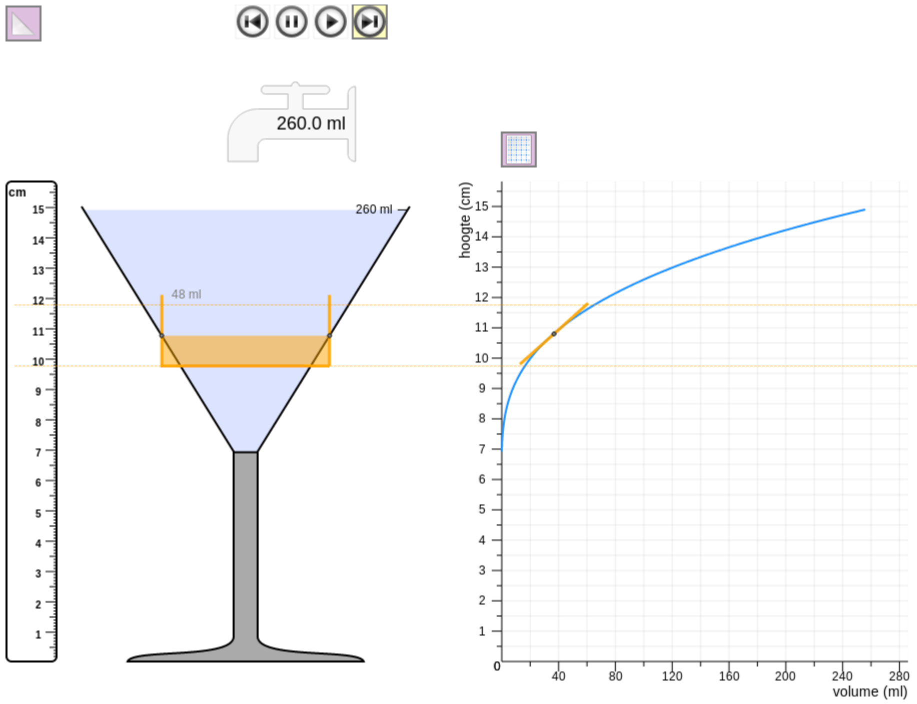


Figure 5.9: Direct link between the imaginary highball glass and its graph as the tangent line to the cocktail glass' curve

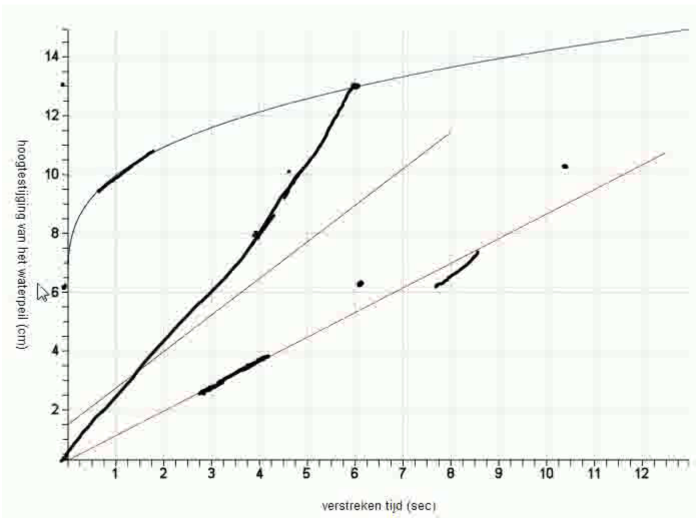


Figure 5.10: When the tangent line is parallel to the highball glass' graph, at that point at the curve, the cocktail glass and the highball glass have the same speed. (Excuses for the bad quality of the image: it is a screen-shot of the interactive white board during the teaching experiment)

```
         the curve, more or less parallel to the brown line)
Teacher: Could you draw that?
Roel:    (draws) This is about, like, about this (draws a short line
         on the brown line and a short line as tangent line to the
         curve)
Teacher: Okay. And what's now the same? Is this brown line and this
         black line (points to the tangent line at the curve)?
Roel:    Then it rises with the same speed (makes with his hands
         parallel a movement from bottom left to top right), the
         lines are directed to the same side.
```

After the tangent line was drawn, the students in this discussion realized that when it was parallel with the highball glass' graph, both graphs had the same steepness and the speeds were the same. For these students, the tangent line became an indication of the steepness and, therefore, the speed.

3. In the last design experiment, after exploring students' suggestions on how to use the highball glass' graph to determine the speed in the cocktail glass' curve, the teacher just introduced the tangent line "as a better way to determine speed". Many students realized that the tangent line was an indication of the steepness of the curve at that point and, connecting that to their understanding of the relationship between the steepness and speed, they realized that the tangent line was also an indication of speed at a point.

In each case, however, once the tangent line was introduced, the students seemed to accept it.

k. *The students came to understand the shape of the graph of a cooling process as being predicted by the steepness of the graph, which depends on the temperature differences.*

In the last design experiment, the students modeled cooling down and warming up of water in a graph. After exploring this context quantitatively, they were not only able to refine that graph to resemble the correct curves, but they also discovered and formulated Newton's Law of Cooling. Thus, when exploring computer drawn graphs of this situation using the tangent-line-tool, from the classroom discussions it appeared that many students came to understand the steepness of the graph to denote the rate of change of the difference between the current temperature and the room temperature.

5.5 *A proposed local instruction theory on teaching instantaneous speed in grade five*

By relating the patterns that emerged from the data, the students' key learning moments, to the emerging LIT throughout our design research process, we are able to formulate the proposed LIT. We present it in two parts: we elaborate students' learning processes when deepening their pre-existing conception of instantaneous speed first, followed by detailing the potential means of support that were necessary for those learning processes to emerge.

5.5.1 *The postulated learning processes*

The context of filling glassware is familiar to the students. All of their lives, they have filled many glasses and bottles of different shapes and sizes. In this context, they have experienced both linear and non-linear situations of change. Their rich intuitive understanding is activated by observing a cocktail glass fill up with a constant influx of water, either an actual glass or in a computer simulation. They immediately realize that the wider the cocktail glass, the more water it takes to have the water level rise with the same amount. They understand that in this situation the water level rises with a continuously decreasing speed; which implies an implicit notion of instantaneous speed. Thus they have a general understanding of the relationship between a glass' width and speed: the wider a glass, the slower the speed; and the smaller a glass, the faster the speed.

Students do have limited experience with interpreting and making graphs. Most graphs they have encountered in primary school have been either bar graphs, straight lines representing linear phenomena, or a graph comprised of straight line-segments representing some more complex phenomena. Their graphing experience does not offer them the tools to sufficiently express their understanding of filling glassware as a continuous non-linear process. However, through a modeling-based learning approach the students are able to develop such a tool: the curve as a Cartesian graph. In a modeling-based learning approach students express their understanding through making models, which then can be discussed, evaluated, and improved. This process of modeling gives the teacher some insight in students' reasoning. We do not presume that the students will come to develop any expert understanding of graphs, but they are able to deepen their preexisting understanding of how to interpret and represent individual point as well as to come to understand the curve to represent the complete continuous process of filling a glass.

When being tasked to try to adequately depict how (fast) the water height is changing in a cocktail glass that is filled with a constant influx, students initially may draw realistic pictures or snapshot models to describe how a cocktail glass fills up. By pushing the students to create more precise models, while having them explore the situation of filling glassware further, the snapshots-model emerge and become taken-as-shared. These snapshots-models can be considered as *models-of* the way the water height in the cocktail glass changes.

Under guidance of the teacher, these models are transformed into graph-like representations the students are familiar with, such as a bar chart or a graph consisting of straight line-segments. Both can be build upon to develop a continuous and curved graph. The straight-lined segmented graph conflicts with students' understanding of the situation as continuously changing. Focusing on what happens in between two points, students come to realize that a straight line between the points is not a good fit, the only alternative that makes sense is a curve between the points. Similarly, the vertical value bars, which have come to signify water heights in the glass at specific moments in time, can be used to build upon. Again, focusing on what happens in between two points, the students understand that more points can be added, and have a sense of how long those bars need to be. The number of points can be varied flexibly in a computer simulation; by increasing that number, the students see the curve come to the fore. Reflecting on continuous graphs of computer simulations and student-generated graphs, and by discussing the relation between the shape of the glass, the rising speed of the water, and the shape of the graph, the students come to see the curve as signifying both the changing value, and the rate of change. As their attention shifts towards the mathematical relations involved—the relations between the character of the change and the shape of the graph, which they develop in the process—, the model starts to become a *model for* more formal mathematical reasoning.

To prepare the students for using the highball glass as a measure for the instantaneous speed in the cocktail glass later on, the focus temporarily has to shift from the cocktail glass to exploring and quantifying speed in the highball glass. The students are extremely familiar with linear situations. They understand that the water level rises with a constant speed in the highball glass and they express that understanding in a graph with one straight line. Although the students are familiar with computing constant speed in a linear situation, that knowledge has to be activated first. Part of understanding a constant speed, which is a ratio, is that it can be expressed using different units, such as m/s or km/h, precisely because it is a ratio. Even then, students' understanding of and skill in computing speed is likely underdeveloped. Through comparing highball glasses of different width, drawing graphs, and computing speeds, the students deepen their understanding of the relationship between a glass' width and the steepness of its graph: the smaller a glass, the faster its speed, and the steeper its graph.

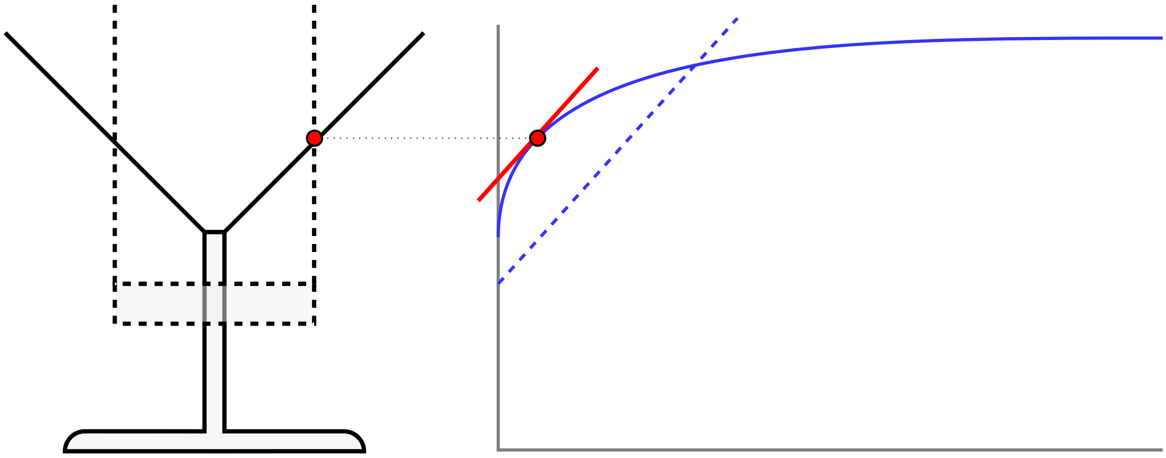


Figure 5.11: The *instantaneous* speed in the cocktail glass at the red point is the same as the *constant* speed in the cylindrical highball glass (dashed line); the highball's graph, the straight dashed line, is the tangent line on the cocktail glass' curve at the red point. (Taken from de Beer, Gravemeijer, and van Eijck (2015))

When asked to determine the speed in the cocktail glass the students may fold back to understanding of computing speed in linear situations, and compute an average speed. Even though this speed does not match their understanding of the continuously changing speed. An obvious improvement that the students probably will come up with, computing the average speed on smaller intervals, does not overcome that mismatch either. We note, however, that when students have developed a deeper understanding of measures of (constant) speed and they have more experience with continuous graphs than the students in our design experiments, they might come up with more fruitful alternative strategies to determine the instantaneous speed.

Nevertheless, to guide the students to invent the highball glass as a measure for instantaneous speed, they have to be asked when the water level rises with the same speed in both the cocktail glass and highball glass. Given their understanding of the relationship between the width and the speed, they immediately realize that both glasses have the same speed when they have the same width. This realization is made tangible by picturing the highball glass inside the cocktail glass. Then, the students come to see that to compute the instantaneous speed at some point in the cocktail glass, they can imagine a highball glass that is as wide as the cocktail glass at that point (see Figure 11, left hand side): the instantaneous speed is equal to the highball glass' constant speed, which they know how to compute. Then the students have constructed the highball glass as a tool to determine instantaneous speed in various glassware.

Next, to guide the students in constructing the tangent line as a tool to determine instantaneous speed in any graph, two avenues of reasoning are explored. The highball glass and tangent line can be directly linked together (Figure 11), making visually explicit the connection between a tool the students already know and a tool that is being introduced. Furthermore, the tangent line, at this moment representing the highball glass' graph, is also an indication for the steepness of the curve at that point. Therefore, in this picture, the tangent line is grounded in the students' understanding of the relationship between a glass' width, the steepness of its graph, and its speed. From here, even after severing the connection between the concrete representation of the glasses and the more abstract representation of their graphs, the students understand that to compute the instantaneous speed at some point on the curve, they can use the tangent line to the curve at that point: the instantaneous speed is equal to the speed indicated by the straight tangent line, which stands for the highball glass' graph, which they know how to compute. As an aside we may point to the similarity with the definition of instantaneous velocity in the fourteenth century by William Heytesbury (Clagett, 1959), which preceded the idea of using the limit to approach the instantaneous speed (Doorman, 2005).

Potentially, the tangent-line-tool also opens the door for students to further develop their conception of instantaneous speed as speed at a moment—at this point instantaneous speed is tightly linked with constant speed, which denoted by a ratio—towards a more general conception of speed as a variable, a rate. By moving the tangent-line-tool over the cocktail glass' curve, students see the tangent line "fall over", from an almost vertical line at the beginning to an almost horizontal line at the end. The simulation of the continuously changing angle of the tangent line depicts instantaneous speed as a variable quantity not unlike the changing water level height in the simulation of filling glassware.

Finally, for the students to deepen and generalize their understanding of instantaneous speed, they have to explore speed in other contexts as well. For a first transitory context, we think cooling down and warming up of water suits well. As with filling glassware, students are familiar with this context because they will have encountered hot or cold beverages, baths, and the like throughout their lives. At the same time, however, temperature and temperature change are not as visible as the rising water level. Nevertheless, they will realize that in both processes, the temperature will approach the room temperature eventually. But how? To find out, students measure the temperature repeatedly to compute the differences in time, temperature, and the average speed on the last measuring interval. Furthermore, they can make predictions based on their measurements and computations. Gradually, they will conjecture that the speed of temperature change is proportional to the difference between the current temperature and the room temperature; they discover Newton's Law of Cooling. They express that understanding with a curve, arguing that, at first, the curve will be steep because the temperature difference is great. Once it has been cooling down for a bit, however, and the difference in temperature has decreased, the curve will flatten out as the speed decreases.

5.5.2 Potential means of support

Besides elaborating students' learning processes in the LIT, attention has to be paid to the means of support in bringing these learning processes to fruit. According to Cobb et al. (in press), part of the argumentative grammar of design research is to identify these means of support and show that these are necessary for these learning processes to occur. We identified three interconnected means of support: the context of filling glassware, the computer simulations, and modeling-based learning.

The context of filling glassware, introduced by Swan (1985) in a secondary-school textbook on functions and graphs, is well-suited for exploring covariation. Not only are students very familiar with it, the context is also simple and can easily be comprehended in its entirety. Unsurprisingly, then, this context has been used more often to explore covariation in primary school and middle school (McCoy et al., 2012; Gravemeijer, 1984-1988), secondary education (Swan, 1985; Castillo-Garsow et al., n.d.; Johnson, 2012), and higher education (Thompson, Byerley, & Hatfield, 2013; Carlson et al., 2002).

What made the context of filling glassware a *necessary* means of support, however, was how it visually connects constant speed, which students already know, to instantaneous speed. In a concrete setting, picturing the highball glass together with the cocktail glass offers a very powerful image. Once students understand the relationship between a glass' width and speed, that image shows where both glasses do have the same width and thus the same speed. More so, it enables the students to invent an imaginary highball glass as a tool to measure the instantaneous speed in any glass (Figure 5.4) that does not need the troublesome limit concept. We are unaware of any other concrete context with a similar salient characteristic.

The context of filling glassware lends itself very well to simulation with a com-

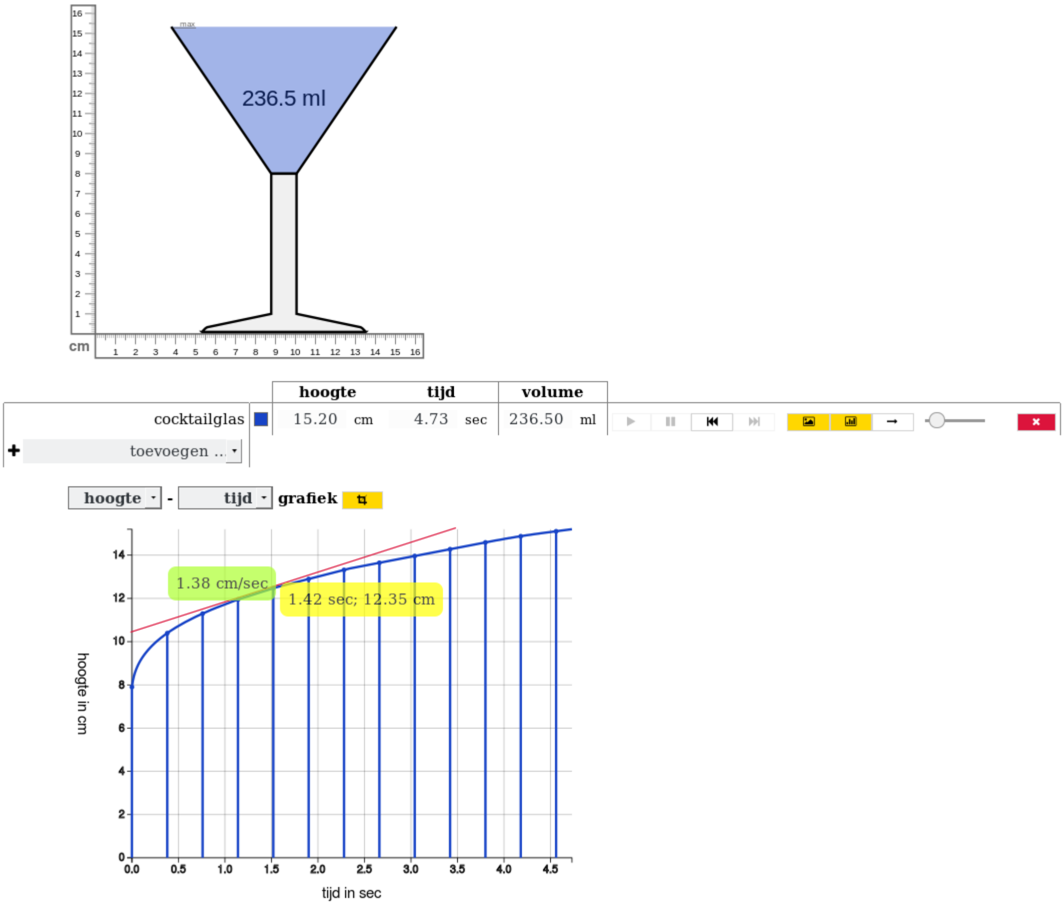


Figure 5.12: Design experiment 3: Computer simulation showing a cocktail glass and both its graph and a bar chart. The interval between the bars can be changed dynamically through the slider on the right. At the point (1.42s; 12.35cm) the tangent-line-tool is used to draw a red tangent line and give its speed (1.38 cm/s)

puter (Figure 5.12). Although one could explore the context in a concrete setting, an interactive computer simulation enables flexible, fast, repeatable, and precise experimentation without creating a wet mess. These characteristics enable an inquiry-based learning approach, such as modeling-based learning, where students and teacher alike can explore the situation safely, formulate and test hypotheses, and discuss, evaluate, and critique their ideas by using the simulation in their arguments. For example, the ability of the computer simulation to draw graphs of any two quantities of water level height, volume, and time, allowed a student in the third design experiment to discover and compute the constant flow rate by exploring the volume/time graph.

Furthermore, the computer simulation could draw both a line graph and a bar chart. This emphasized and supported the dual understanding of graphs we wanted the students to construct: both as a way to reason about individual data points as well as to reason about the whole continuous process. Moreover, by increasing the number of bars shown, slowly the shape of the curve comes to the fore, offering the students a powerful image and support them in accepting the curve as a good fit to describe their understanding of the situation. Moreover, it creates an affordance for shuttling back and forth between a continuous and a discrete image; where the difference between the two adjacent bars offers a complementary entry to the instantaneous speed in a given point. In a similar vein, once the students construed the highball glass as a measure for instantaneous speed, the highball-glass-tool not only allowed them to put their invention to the test, it also made their theory tangible. Finally, the tangent-line-tool allowed students to quantitatively explore filling glassware as well as other contexts. More interestingly, however, by moving the mouse cursor over the curve using the tangent-line-tool, the changing speed is aptly visualized, strengthening students' understanding of the relationship between speed and steepness of a curve. Furthermore, this simulation of the changing angle of the tangent line over time offers a potential avenue to support students in developing an understanding of instantaneous speed as a variable quantity in itself.

The last means of support we identified as necessary is modeling-based learning. Before we introduced the modeling-based learning approach in design experiment 2, however, we had trouble interpreting students' understanding of and skill with graphs and there was a lack of conceptual discussion about speed. To overcome that problem, we applied the modeling-based learning approach in the next two design experiments. We had the students express their understanding through modeling and these expressed models were presented, discussed, and evaluated in class, allowing the students to improve their models. It also gave the teacher indirect access to their mental models, allowing him to better support students in constructing a deeper understanding of graphs in relation to speed. For example, in design experiment 3 the modeling-based learning approach enabled the students in one classroom to discover the curve on their own, while it prepared the other classroom to accept the curve as a model fitting their understanding of filling glassware.

We like to emphasize that these three means of support were part of a larger learning environment wherein the students' learning process took shape. The teacher plays an important role in shaping that environment to create a suitable classroom culture with appropriate social norms. Applying a modeling-based learning approach or using flexible tools such as the computer simulation to its fullest potential is not straightforward.

5.6 Conclusions and discussion

In this chapter we presented the results of a series of teaching experiments that were conducted to design, try out, and improve a LIT on teaching instantaneous speed in grade five. In a retrospective analysis, looking for patterns in the whole data set, encompassing all experiments, we identified a set of key learning moment of the students, which allowed us to construe a potentially viable LIT.

Our approach was based on Stroup's (2002) "qualitative calculus" approach to deepen students' intuitive notions of speed. Stroup argues that qualitative understanding of calculus-like concepts is a worthwhile enterprise in itself and not only a transformational phase towards more conventional ratio-based understandings of rate (Stroup, 2002). We found it a good foundation to build on when supporting students in constructing also a quantitative understanding of instantaneous speed that is not based on taking the limit of average speed. Despite the tentative nature of the LIT, we may argue that it offers a potential solution to the troublesome limit concept (Tall, 1993) when learning about instantaneous speed. Although circumventing this difficult concept in our LIT, we do acknowledge that gaining understanding of a threshold concept is of fundamental importance to being able to deepen one's understanding of advanced concepts that build on that understanding (Perkins, 2006). In the case of limit, however, this has to come later. However, deepening and expanding the students' understanding of instantaneous speed in primary school will provide the students with a strong conceptual base for coming to understand the limit concept.

Like most approaches exploring teaching calculus-like topics before high school, we also used computer simulations and graphs in our approach. Unique to our approach to teaching instantaneous speed, however, is that we circumvent the limit concept also while supporting students to quantify instantaneous speed. We try to support students in expanding their intuitive understanding of instantaneous speed by having them compare constant speed with instantaneous speed in a computer simulation. Likewise, average speed takes a less prominent place here than in conventional approaches to instantaneous speed. This is fortunate because students of this age do have a limited understanding of average speed, even though it is part of their curriculum. More so, in the context of filling glassware, the tangibility of the relationship between the constant speed in a highball glass and the instantaneous speed in a cocktail glass allows students to develop a quantitative measure of instantaneous speed that relies on their understanding of constant speed.

Students are able to translate this understanding to graphs. Despite their limited experience with and skill in drawing and interpreting graphs, through a modeling-based learning approach the students were able to come to see the curve as a model that fit their intuitive understanding of instantaneous speed. It describes the pattern of the continuous changing speed, thereby denoting the instantaneous speed. By linking the highball glass' graph to the tangent line at the cocktail glass' graph, a curve, the students came to accept the tangent line as a quantitative measure of instantaneous speed. From there, the students were able to shift their understanding to the context of warming and cooling as well, suggesting they developed a more general understanding of instantaneous speed and graphs. From an emergent modeling (Gravemeijer & Doorman, 1999; Gravemeijer, 1999) perspective, the students' learning process can be seen as a development of the virtual highball glass as model of instantaneous speed in the context of filling glassware to the tangent line as a model for instantaneous speed for any dynamic situation represented by a line graph.

We like to reiterate that our proposed LIT is not intended as a ready-made product, but as a potential viable theory for other researchers, educational designers, and maybe teachers to develop a learning trajectory on instantaneous speed adapted to their situation. For the LIT to be potentially viable, it should be transferable (Smaling, 2003), which means that the following two requirements have to be met: First, from the experience and perspective of potential users of the LIT, the students' learning processes described in the LIT should be plausible in the context of the teaching experiments; Second, the potential users of the LIT should be able to ascertain the potential for adaptation of the LIT to *their* (practical) situation. In this regard, we see potential problematic areas of our research as a framework delineating further development and adaptation of the proposed LIT.

For example, given a possible lack of teachers' affinity and experience with (teaching) STEM and instantaneous speed in particular, it might be unrealistic to expect teachers to adapt the proposed LIT themselves. However, we do think it is a solid foundation to develop materials that can support teachers in exploring instantaneous speed in their classrooms and gain more experience teaching STEM. Similarly, although the modeling-based learning approach is quite different from more conventional instructional approaches most teachers will be familiar with, the proposed LIT offers a meaningful context to explore modeling-based learning as an opportunity for professional development for teachers. Finally, with respect to the lack of students' expertise with graphing and their limited command of speed in linear situations, a more thorough treatment of graphing and speed seems to be in order. The proposed LIT can serve as a starting point to improve the current curriculum in this regard. In any case, we envision a collaboration between researchers, educational designers, and teachers to further explore teaching instantaneous speed in fifth grade.

6

Conclusions and discussion

The aim of this dissertation was to investigate how 5^{th} grade students might be taught about instantaneous speed. To answer this question a design research approach was followed, which was executed in a series of three design experiments, and was preceded by a small number of one-on-one teaching experiments. All of which was reported upon in the preceding chapters. However, instead of giving a continuous narrative detailing this design research, this dissertation focused on four separate studies that emerged from the design research project. In this chapter, a summary of these studies is given, followed by a discussion of the over-all findings.

6.1 Summary

6.1.1 Introduction

The aims and scope of this thesis are elucidated in Chapter 1. In answer to the call for a new STEM education in primary school for the 21^{st} century (Léna, 2006; Gravemeijer & Eerde, 2009; Gravemeijer, 2013; van Keulen, 2009; Millar & Osborne, 1998)—innovative STEM education should be inquiry-based, ICT-rich, close to students' world-view, and able to integrate in the primary school curriculum—, a design research project was started to answer the research question *How to teach instantaneous speed in 5^{th} grade?* The topic of instantaneous speed was chosen because the interpretation, representation, and manipulation of dynamic phenomena are becoming key activities in the information society and is firmly rooted in the realm of STEM and key to a better understanding of dynamic phenomena is the concept of instantaneous rate of change. In the context of primary school the term "speed" is preferred because students of this age will not have developed the level of abstraction associated with the use of "rate of change" in the literature.

Design research is well-suited to explore how to teach topics in earlier grade levels than usual (Kelly, 2013), and instantaneous speed is conventionally taught first in a calculus course, which is not part of the primary school curriculum. Design research is a reaction to a perceived gap between educational practice and research (The Design-Based Research Collective, 2003; van Eerde, 2013; Reeves et al., 2010), which is to be bridged by exploring theoretical issues regarding students' learning processes through studying students' *actual* learning processes as they happen in a real classroom instead of in a laboratory setting (Collins et al., 2004). It is an interventionist process of iterative refinement that takes real-world classroom practice into account to create some instructional artifact and a theory on how that artifact works (Cobb, 2003; The Design-Based Research Collective, 2003; Barab & Squire, 2004; Reimann, 2011; Plomp, 2013; Bakker & van Eerde, 2013). It further aims at exploring theoretical issues that go beyond the scope of a particular design research project (Barab & Squire, 2004).

In this thesis, the approach to design research outlined in Gravemeijer and Cobb (2013) was applied to develop a local instruction theory (LIT) on teaching instantaneous speed in grade five. A design research project starts by formulating the learning goals, the instructional starting points, and an initial LIT, based on the literature and any other source that might contribute to the researcher's understanding of students' prior conceptions (Chapter 2). These other sources, such as text books, teacher guides, prior experiences, and basically everything deemed useful by the researchers, feature strongly in the starting-up phase because design research is often applied in areas without much relevant literature (Gravemeijer & Cobb, 2013).

Once formulated, the LIT is elaborated, adapted, and refined in multiple design experiments (Cobb, 2003; Gravemeijer & Cobb, 2013) (Chapter 4). A design experiment consists of three phases:

1. The LIT is (re)formulated and, based on that LIT and any other sources

deemed relevant by the researchers, an instructional design is developed.

2. That design is tried and adapted in a teaching experiment consisting of a sequence of micro design-cycles of (re)developing, testing, and evaluating instructional activities and materials.

3. During the retrospective analysis (Chapter 3), an analysis of what happened during the teaching experiments informs the refinement of the design, and an analysis why this happened informs the refinement of the LIT.

This refined LIT is the starting point of the next design experiment. This process yields both a working prototype and a theory on how that prototype works. In line with the theory-driven nature of design research the focus in this thesis is on the LIT rather than on the prototype. The proposed LIT is presented in Chapter 5.

The design research project consisted of four partial projects: a starting-up phase followed by three subsequent design experiments. As design research is a process of iterative refinement, experiences gained and results found in earlier partial projects are reflected and apparent in the 3rd design experiment. As a result, this thesis focuses on this last design experiment and its substantive relationships with the preceding partial projects. This thesis is not set up as a monograph offering a continuous narrative detailing the design research project. Instead a format is chosen that consists of four separate studies that did emerge from the design research project:

- In the starting-up phase, 5th grade students' level of covariational reasoning was investigated by conducting a one-on-one teaching experiment with nine 5th grade students and analyzing their behavior using the covariation framework (Carlson et al., 2002).

- The generation of new theory in design research through abductive reasoning was explored in the context of the retrospective analysis of the third design experiment.

- In an effort to contribute to the codification of educational design, the researchers' own learning process was discussed as an example of how educational-design knowledge and practices might be documented.

- Finally, the proposed LIT was formulated in terms of students' key learning moments that emerged from the data collected during the whole design research project.

These studies are discussed next. Furthermore, to get a complete overview of the design research project, after discussing the starting-up phase, a short overview of the three design experiments is included as well.

6.1.2 Starting up the design research project

The starting-up phase is described in Chapter 2. To start up the design research, two avenues were taken to explore 5th graders' prior conceptions of speed, a literature review and a small-scale study, both of which are discussed next. Finally, the first ideas regarding a LIT on instantaneous speed are formulated.

Literature review on students' conceptions of speed and teaching calculus-like topics early

In both the literature on primary school students' conceptions of speed and the primary school curriculum, speed is treated as average speed and interpreted as a ratio of distance and time. In general, primary school students have problems with relating distance and time to speed (Groves & Doig, 2003), and 5th grade students have trouble developing an understanding of speed as a rate (Thompson, 1994b). Stroup (2002) attributes these problems to conventional approaches to teaching speed that favor ratio-based understanding of speed over students' intuitive understanding. Instead, he argues for developing students' qualitative understanding of speed. This "qualitative calculus" approach deviates further from conventional approaches by starting with exploring non-linear situations of change (Stroup, 2002). In most other studies on primary school students' conceptions of speed instantaneous characteristics have not been explored. Because instantaneous speed is usually taught first in a calculus course, the literature review was extended to teaching calculus-like topics early on in the mathematics curriculum. Studies on teaching calculus-like topics in primary school (Nemirovsky, 1993; Thompson, 1994b; Boyd & Rubin, 1996; Nemirovsky et al., 1998; Noble et al., 2001; Stroup, 2002; Ebersbach & Wilkening, 2007; van Galen & Gravemeijer, 2010; Kaput & Schorr, 2007) share two characteristics: computer simulations and Cartesian graphs.

Research suggest that young students are able to explore speed mathematically using computer simulations, which are a natural fit to explore dynamic phenomena. Computer technology enables inquiry-based learning approaches because it allows students to explore more authentic, realistic, and complex problems (Chang, 2012; Ainley et al., 2001). However, because they can function as black boxes, Gravemeijer et al. (2000) argue that computer simulation only induce adequate learning processes when embedded in a suitable instructional sequence. Furthermore, if students are sufficiently supported in using graphs, they appear to be able to express their understanding through graphs, even when they lack graphing experience (Leinhardt et al., 1990). More specifically, social-cultural approaches to graphing (Roth & McGinn, 1997) and (guided) reinvention approaches (diSessa, 1991) have been proposed.

Although the results of the literature review did not offer sufficient information to formulate an initial LIT for teaching instantaneous speed, it did offer enough basis for developing and performing a small-scale study to explore 5th graders' understanding of speed in situations with two co-varying quantities.

A small-scale study to explore 5th graders' understanding of speed

To explore 5th graders' understanding of speed in a situation with two co-varying quantities in the context of filling glassware, a short instructional sequence and computer simulation was developed. The students were asked to perform similar activities with a highball glass, a cocktail glass, and an Erlenmeyer flask:

1. turn the glass into a measuring cup by dragging hash marks to the right place on the glass
2. evaluate the solution
3. draw a graph of filling that glass
4. and evaluate the graph by comparing it with the computer-drawn graph.

Eight one-on-one teaching experiments (Steffe & Thompson, 2000) with above average performing 5th graders were performed. The videos of the experiments were transcribed and, together with the screen-captured computer sessions, analyzed using the covariation framework (Carlson et al., 2002). Students' utterances were coded for one of five developmental levels of covariational reasoning defined in the covariation framework. These levels ranged from a basic understanding of co-varying quantities to an understanding of a dynamic situation in terms of instantaneous speed. It was found that the students came to reason at levels two and three: they were quite capable in estimating the relative amount of change at certain points or intervals. The students almost never talked about (instantaneous) speed, however, and when they did it had to be classified as pseudo-behavior (Vinner, 1997).

However, it was also concluded that the covariation framework did not offer the grain size nor the focus on instantaneous aspects of change to be a good instrument to study 5th grade students' conceptions of instantaneous speed. A serious limitation of the usefulness of the covariation framework in primary school appears to be in the limited vocabulary and graphing skills of the primary school students. Conversely, the investigation with the framework was valuable in that it pointed to the importance of graphing skills and the fact that 5th grade students' skills were insufficient.

Starting points for an initial LIT

Nevertheless, the experiences during the one-on-one teaching experiments and the findings of the literature review together allowed for the formulation of starting points for the initial LIT. This initial LIT aimed at expanding on Stroup (2002)'s qualitative calculus approach by supporting 5th graders in developing both a qualitative and a quantitative understanding of speed in the context of filling glassware, based on the following line of reasoning:

Assuming that students' intuitive notion of instantaneous speed is close to a historical notion of instantaneous velocity that did not build on the complex process of taking the limit of average speeds, students might come to equate the *instantaneous* rising speed in the cocktail glass with the *constant* rising speed in an (imaginary) highball glass. After exploring the relation between a glass' shape and the rising speed, students may realize that the instantaneous rising speed in the cocktail glass at a given height is equal to the constant rising speed in an highball glass that has the same width as the cocktail glass has at that height. This would eventually enable students to quantify the instantaneous rising speed in a point by computing the constant rising speed of the corresponding highball glass. Building on the correspondence between a highball glass's graph and the steepness in a point of a curve, the students may construe the tangent line in a point of a curve as an indicator for the instantaneous speed in that point. Given students' familiarity with constant speed and graphing linear situations, students then may quantify the instantaneous speed by computing the rise over run of the tangent line.

From the one-on-one teaching experiments it was learned that conceiving filling a highball glass as a linear process was self-evident for the students. This was also the case for the linearity of the graph. The cocktail glass revealed that they easily fell for the linearity illusion, which they would overcome, however, when they saw the simulation. Important was that the students showed to understand the mechanism; they realized that the rising water level in the cocktail glass slows down, because the glass gets wider. Graphing proved ambivalent, they could not draw the correct graph, but they nevertheless seemed to appreciate the curve—even though they could not explain it.

6.1.3 Overview of the design experiments

Design experiment 1

The first design experiment revolved around an instructional sequence that started with students making measuring cups from a highball glass, a cocktail glass, and an Erlenmeyer flask in a computer simulation. The second lesson was dedicated to exploring speed in the highball glass through drawing its graph and computing its speed. Only then, in the third lesson, would the attention shift to the problem of instantaneous speed. It was expected that, once the students had realized that the speed in a cocktail glass and a highball glass would be the same when the widths are the same, would come to see that they could determine the instantaneous speed at a specific point in the cocktail glass by computing the constant speed of a (computer-drawn) virtual highball glass with the same width as the cocktail glass at that point. Next, by directly linking the virtual highball glass with its graph as a tangent line on the cocktail glass' curve, the students were expected to come to accept the tangent line as a tool to measure instantaneous speed in a graph. It was anticipated that this would enable them to reason about speed in other contexts as well.

Through Design Experiment 1, it became even more clear that the power of the context of filling glassware lies in the fact that it offers the students a powerful theory to reason about the covariation process on the basis of their understanding of the relationship between a glass' width and its speed. It was further learned that the students had no problem with answering the question, "When is the speed in the cocktail glass equal to the speed in the highball glass?" This answer presupposes that one thinks of speed in a point. The students could handle the tangent-line-tool, but their level of understanding was doubted. Further, they still did not manage to come up with a continuous graph; it is believed that starting with the highball glass might have put them on the wrong track.

Design experiment 2

In design experiment 2, to overcome the problem of a discrete learning environment, it was decided to remove the measuring cup activities. Furthermore, the students' learning process was to revolve around the non-linear situation of filling the cocktail glass. To increase conceptual discussion about speed a modeling-based learning (MBL) approach was selected to allow students to express their understanding more explicitly while presenting, discussing, and evaluating their models in class.

The instructional sequence started by asking the students to model filling a cocktail glass, and to improve their models after exploring the situation in a computer simulation. They were not expected to start using Cartesian graphs, but once the graph was introduced by the teacher, it was expected that they would see its value and be able to use it to predict the water level height at any moment and to connect its shape to their image of filling glassware. They would extend their understanding of the relationship between a glass' width and speed to include the steepness of the curve. After a brief exploration of computing constant speed in the highball glass, the students would be guided towards construing the highball glass as a tool for measuring instantaneous speed by exploring when the speed is the same in both glasses. The students were expected to come to see that the cocktail glass' curve is as steep as the highball glass' straight line in the point they have the same width. Finally, moving to the context of toy-car racing, the students were expected to explain a race given a graph, indicate where the car went fastest or slowest, and use the tangent line to quantify its speed.

The (small scale) Design Experiment 2 taught the value of MBL; the modeling activity did indeed help to make the students' thinking visible and topic of discussion. Again the students' lack of understanding of, and fluency with, measures of speed revealed itself. Again there was no success in developing continuous graphs, which could suggest that maybe students of this age are chunky thinkers (Castillo-Garsow, 2012). Nevertheless, the students construed the tangent line to the cocktail glass' curve parallel to the graph of the highball glass as an indicator of the speed in a given point.

Design experiment 3

Design Experiment 3 was a turning point because the students in one of the two classrooms showed that they were able to invent a continuous graph by themselves. The catalyst proved to be a critical reflection on the shape of the segmented-line graph, while using their understanding of the relationship between a glass' width and speed, and thus of the character of the covariation. Thus, they were not chunky thinkers; they were continuous thinkers (Castillo-Garsow, 2012) who have difficulty with graphing. In addition, it showed that there is another road to the continuous graph, which was followed by the students in the other classroom where they did not have the benefit of a segmented-line graph. These students came to understand the continuous graph via shrinking the intervals of a bar chart. Next the students construed the tangent line as an indicator of the speed at a given point, by drawing a line parallel to the linear graph of the highball glass. It was further learned that the students could build on those ideas to come to grips with the shape of the graphs of a cooling process by assuming that the tangent line in a given point depends on the difference between the actual temperature and the final temperature.

However, even though the students had a tool to measure instantaneous speeds, they did not yet develop a sound understanding of how to quantify speed. They were hampered by the fact that they did not have a sound basis for calculating speeds; they clearly needed more experience with quantifying constant speeds in a variety of ways (with different units), and relating those with the corresponding graphs.

6.1.4 The generation of new explanatory conjectures in design research

Chapter 3 focuses on one of the key aspects of design research: the generation of new theory during the retrospective analysis. It is shown how a process of abductive reasoning during the retrospective analysis of the third design experiment led to the generation of the conjecture that primary school students come to the classroom with a continuous conception of speed and only switch to discrete reasoning because of a lack of means for visualizing continuous change. This, in turn, led to the realization that average rate of change is a hindrance rather than a necessity in teaching instantaneous rate of change in primary school.

Abductive reasoning is triggered by an unexpected event (Fann, 1970) during the teaching experiments. These surprising facts are treated as an indication of a misalignment between the researcher's understanding of (anticipated) students' learning processes and their *actual* learning processes. Abductive reasoning tries to resolve this conflict by generating new explanatory conjectures. Subsequently, the data collection is re-examined to determine the extent to which these conjectures are supported or have to be rejected. In Chapter 3 it is illuminated how abduction plays a specific role in design research. To that end, the teaching experiment leading up to the unexpected event is described in detail, followed by a two-step retrospective analysis.

Sketching the unexpected event and the context leading up to it

The third design experiment started by reformulating the LIT based on the findings of the previous design experiments. Based on this LIT a four-lesson instructional sequence was developed around this LIT, which was tested in two gifted 4^{th}-6^{th} grade classrooms taught by the same teacher. During the second lesson, a surprising difference between what happened in the two classrooms resulted in process of abductive reasoning in the retrospective analysis.

In the first lesson, the students modeled filling a cocktail glass four times to allow for an iterative improvement. During these activities, although many of the first models were quite realistic depictions of the situation, the discrete snapshots model became taken-as-shared in both classrooms. At the start of the second lesson the students were asked to create a minimalist model based on an earlier model. Most of these minimalist models were discrete representations, but in each classroom there was one graph-like model with continuous characteristics. At this point, what happened in both classrooms deviated: in classroom 1 (C1) this continuous graph-like model was discussed in class, in classroom 2 (C2) it was not.

C2 followed the anticipated learning trajectory. Under the guidance of the teacher, the students condensed the discrete minimal models into a bar graph first. The bars came to signify water heights in the glass at specific moments in time. Next, these bars were connected by arrows, signifying the change between the bars. Finally, after introducing the computer-drawn curve, the students were expected to come to see the curve as signifying both the changing value and the speed. In C1, however, the students' actual learning trajectory was different. While discussing the student-drawn graph-like minimalist model, which was a segmented straight-line graph, the students argued that the straight line segment had to be a curve: they invented the curve by themselves. This unexpected event led to a process of abductive reasoning during the retrospective analysis.

Retrospective analysis

The retrospective analysis followed a two-step method based on Glaser and Strauss's (1967) comparative method, in particular, the elaboration of Cobb and Whitenack (1996) on this method was used:

1. conjectures about *What happened?* were formulated and tested against the data. This process resulted in six conjectures.

2. conjectures about *Why did this happen?* are formulated and tested against the data, which resulted in the generation of the following new explanatory conjecture: The students come to the classroom with a continuous conception of speed. They only switch to discrete reasoning because of a lack of means for visualizing continuous change.

This retrospective analysis showed that students' reasoning was grounded in continuous reasoning, while discrete reasoning functioned as a tool to get a handle on continuous processes. Furthermore, students easily reasoned about constantly changing speed, which implies a conception of instantaneous speed. This observation triggered a process of abduction at the design research level, which generated the question: "Do the students ever use average speed?" With one exception, it never showed that the students were thinking of average speed. This stands in sharp contrast with the common practice of starting instruction on speed by introducing average speed. Starting with average speed is problematic: it promotes discrete thinking and could be the source of Castillo-Garsow (2012) problematic chunky thinkers. It would make more sense to explore constant speed, which subsequently can be connected to the students' notion of instantaneous speed.

Generalizability and the generating new ideas in design research

Mark that the generalization of these findings are possibly limited by the uniqueness of the classroom situation—it was tried in two mixed gifted 4th-6th grade classrooms, after all—and the fact that the line graph was introduced by students in C1 seemed more a lucky accident than a controlled act on behalf of the teacher. On the other hand, once the line graph was presented, the teacher did recognize its didactical value and was able to guide the discussion successfully to have students deepen their understanding. In a sense, the teacher was the perfect match: He was one of few primary school teachers who had followed a calculus course in high school. It cannot be expected from the average teacher to have as much insight and experience with calculus-related topics and graphs: these topics are simply not part of the primary school curriculum.

Admittedly abduction does not offer the same rigor as deduction and induction. However, the primary goal of design research is to find out how things work, not to establish for a fact how things are. By being explicit about the abductive argument underlying the development of the LIT, special attention is paid to the justifications common to design research, such as ecological validity, trackability (Smaling, 1990), process oriented causality (Maxwell, 2004) and consilience (Gould, 2011). With respect to abductive reasoning, the unexpected events that trigger it are an indication that more attention should be paid to letting the object speak (Smaling, 1992) which might reveal a clear misalignment between researchers' prior understanding of students' learning processes and the actual learning processes. To resolve this distortion, the researchers have to re-examine their prior conceptions while considering students' perspective more strongly and formulate new explanatory conjectures. Subsequently, if these new conjectures can be grounded in the data collected, and are added to the

LIT, the LIT itself becomes more objective as it does do better justice to the students' learning process.

6.1.5 Design research as an augmented form of educational design

Design research builds on educational design to create both a product and a theory detailing how that product works. Due to the theory-driven nature of design research, that product often does not evolve beyond a prototype (Burkhardt & Schoenfeld, 2003), which limits the utility of the theory. Schoenfeld (2009) recommends project teams that balance expertise in design and research, but that might not be a realistic option for many design research projects. This suggests that researchers have to acquire practical design skills, but as the educational design community lacks both an institutionalized form of schooling and professional literature (Schunn, 2008), this is not straightforward. To overcome this problem, codification of design practices has been proposed (Schoenfeld, 2009), to which design research can play a role.

To develop theory implies a commitment to strengthen the credibility of the theory by justifying its claims and a commitment to allow other researchers to assess the trustworthiness of the process leading up to those claims. The latter can be satisfied by enabling outsiders to retrace the process by which those claims are produced (Smaling, 1990), which means to give a detailed account of the design research process and the researcher's own learning process embedded in it. It is argued that this method of reporting on the learning process of the researchers can function as a paradigm for the way educational designers might want to document their practices and knowledge. To offer an example, the researcher's own learning process is elaborated in Chapter 4 by tracking the development of the instructional sequence from the starting-up phase through the three subsequent design experiments. Both the starting-up phase and the three design experiments have been summarized in a previous section.

Empirically-grounded theory and educational design

The development of the prototypical instructional sequence is illustrated in terms of the researcher's own learning process, which encompassed both the design decisions and the rationale for those decisions. They offer a framework of reference on the basis of which teachers may adapt the instructional sequence to their own classroom and they offer support for instruction design and further theory development. Such a framework of reference may take the form of a LIT that offers a rationale for the prototypical instructional sequence that is developed alongside the LIT. In this manner, design research offers a different kind of support for teachers than most textbooks do.

Apart from offering an example of documenting instructional design decisions and practices, Chapter 4 also elaborates on what makes design research credible, even though it does not follow the classical research method of an (quasi-)experimental design. These theoretical findings can be substantiated by the virtual repeatability of one's research by other researchers (Smaling, 1990). The goal of design research is to generate a theory on how the intervention works. This kind of research employs a process-oriented perspective on causality, stating that, in principle, causal claims could be based on a single case. When aiming at LITs, a single case is the classroom as a whole. To justify that there is a causal relation two methods are used in design research: validating existing conjectures and generating new explanatory conjectures.

Conjectures that are confirmed by the students' actual learning process remain part of the LIT and are tried and refined again in the next design experiment. As a result, conjectures are confirmed or rejected in multiple different situations, offering a form of triangulation that adds to the understanding of students' learning processes in terms of these conjectures. Those observations enable to develop some theories about the mechanisms that were at play here. In addition to this, new explanatory conjectures are generated through abductive reasoning which could be tested on the available data. However, only claims about the students who participated in the experiments can be made and the claims have to be grounded carefully in the observational data to make sure that the conclusions are valid for the majority of the students in the teaching experiments. It was judged that this was predominantly the case in the classrooms where the experiments were carried out. In addition, one usually also carries out a retrospective analysis, which results limit themselves to statements about the actual design experiments. However, a design experiment can be treated as a paradigm case. The goal then is to come to understand (the role of) the specific characteristics of the investigated learning ecology in order to develop theoretical tools that make it possible to come to grips with the same phenomenon in other learning ecologies. The LIT offers a theory of how the intervention works, which teachers and instructional designers can adjust and adapt.

To develop theory implies a commitment to the credibility by justifying the theoretical claims and a commitment to allow other researchers to assess the trustworthiness of the process leading up to those claims. The latter commitment can be satisfied by trackability (Smaling, 1990), which means giving a detailed account of the design research process and the researchers' own learning process embedded in it. This will encourage design researchers to pay attention to practical issues of design and will allow design researchers to create theories that are more practical applicable. Design research can also be taken as a paradigm that may show educational designers the value of documenting design decisions and anticipated students' learning processes. Design research can be seen as an augmented form of educational design, which offers educational designers indications on how to handle the issue of documenting their practices and knowledge.

6.1.6 A proposed LIT on teaching instantaneous speed in 5th grade

By capitalizing on the results of the various design experiments a LIT on teaching instantaneous speed in 5th grade is proposed in Chapter 5. This chapter starts by summarizing the theoretical background and detailing the methodology used in this research project: design research. More specifically, the theoretical underpinnings of the LIT are elaborated in terms of the three instructional design heuristics of Realistic Mathematics Education: guided (re)invention, didactical phenomenology, and emergent modeling. The proposed LIT is based on the patterns in students' learning processes that were identified in the data of the various design experiments. This allowed for a triangulation on two levels. At the level of a single design experiment a multitude of data is collected and used to validate and generate conjectures, and these conjectures are validated or refuted in multiple design experiments, strengthening their empirical basis.

Reporting on the results, basically Cobb et. al.'s (in press) recommendation for an argumentative grammar for design experiments was followed, which requires the justification of the theoretical findings of a design experiment by a) showing that the students' learning process is due to their participation in the design experiment, b) describing that learning process, and c) enumerating the necessary means of support for that learning process to occur. The first requirement is self-evident as 5th graders are not taught on this topic. The other two requirements are split into a documentation of the key learning processes, and a separate description of the envisioned learning process and the means of support that learning process.

Students' key learning moments

The patterns that emerged with respect to students' key learning moments are put in context by a description of students' instructional starting points. These starting points were the same in each teaching experiment: the students had limited graphing experience, they had trouble computing speeds, and they were thinking in terms of instantaneous speed from the start. There is no indication that the situation will be much different in other classrooms. Given these starting points, several key learning moments were identified in the data. Notably, the students were familiar with linearity, but broke through the linearity illusion (de Bock et al., 2002) easily when seeing the cocktail glass fill up. They understood the relationship between a glass width and its speed, allowing them to realize when the speed in the cocktail glass and highball glass is the same. Despite the students' limited graphing experience, once the curve was introduced—in one classroom the students even invented it themselves—they accepted it as a better model than the discrete snapshots models they had created earlier. They were able to construe the tangent line as an indicator of the speed in a given point by combining the cocktail glass' graph and the graph of a highball glass.

Proposed LIT

Based on these key learning moments, the proposed LIT is formulated in terms of the postulated students' learning processes and the potential means of support necessary for those learning processes to emerge. The students' learning processes may be summarized as follows:

Given a cocktail glass, students are given the task to make a drawing of how the water height changes when the glass fills up. After observing it fill up, they notice that the water level rises slower and slower, and they realize this is the result of the glass' increasing width. This realization allows the students to form valid expectations about the process of filling glassware and they come to depict it both as a discrete bar chart as well as a continuous graph. It is expected that the students link the curve of the continuous graph with the continuous change of the speed of the rising water: at every moment that speed is different. From this perspective, students come to interpret the speed of rising as an instantaneous speed.

That conception is deepened both qualitatively and quantitatively by exploring two avenues of thought. First, by comparing the speed in the cocktail glass with the constant speed in a cylindrical highball glass in order to answer the question when the water rises with the same speed in both glasses. The constant speed of an imaginary highball glass becomes a measure for the instantaneous speed in the cocktail glass. Second, building on that understanding, trying to measure speed in a graph by interpreting the straight line graph of the highball glass as a tangent line on the curve of the cocktail glass. Throughout this process, the representations of the speed in the highball glass act as an emergent model of measuring instantaneous speed. Finally, students' understanding of speed in terms of graphs and tangent-line can be translated to other contexts as well.

For this learning process to emerge, three potential necessary means of support are identified:

1. The context of filling glassware is well-suited to explore covariation because it is very familiar to students and it visually connects constant speed (represented by a highball glass) to instantaneous speed (in a cocktail glass). In a concrete setting, picturing the highball glass together with the cocktail glass offers a very powerful image that allows students to construe a measure for instantaneous speed based on constant speed. In other contexts, this is not straightforward.

2. Computer simulations combined with graphs supported the students in developing both a discrete and continuous understanding of graphs despite students' limited graphing experience. Furthermore, the highball-glass-tool allowed the students to put their invention of the highball glass as measure for instantaneous speed to the test, making it more tangible. Finally, the tangent-line-tool allowed students to quantitatively explore filling glassware as well as other contexts. It allowed an inquiry-based learning approach.

3. The modeling-based learning approach not only gave the teacher indirect access to the students' mental models, it also allowed him to better support students in constructing a deeper understanding of graphs in relation to speed. In one classroom it even enabled students to reinvent the curve.

Characterizing the proposed LIT and its application

Unique to the presented approach to teaching instantaneous speed is that it circumvent the troublesome limit concept while supporting students to coming to understand an quantify instantaneous speed. Students are supported in expanding their intuitive understanding of instantaneous speed by having them compare constant speed with instantaneous speed in a computer simulation. This means that compared to conventional approaches, average speed plays a minimal role. Furthermore, the tangibility of the relationship between the constant speed in a highball glass and the instantaneous speed in a cocktail glass allows students to develop a quantitative measure of instantaneous speed that relies on their understanding of constant speed. For, despite their limited experience with graphs, students were able to translate this understanding to graphs, accepting the curve as a fitting model for their intuitive understanding of instantaneous speed. By linking the highball glass' graph to the tangent line at the cocktail glass' graph, a curve, the students came to accept the tangent line as a quantitative measure of instantaneous speed. From there, the students were able to shift their understanding to the context of warming and cooling as well, suggesting they developed a more general understanding of instantaneous speed and graphs.

Although the proposed LIT is considered a potential viable theory, it clearly cannot easily be used in regular classrooms. Apart from the fact that the corresponding instructional sequence has to be elaborated to cater for the students' limited graphing abilities and limited understanding of quantifying constant speed, there are issues of teacher professional development. For others to adapt it to their own situation, the proposed LIT should be transferable (Smaling, 2003). This means that the findings should both be plausible for others and that they should be able to ascertain the potential applicability to their situation. Some aspects of the proposed LIT are problematic in this regard, implying potential avenues for further development and research. Therefore, a collaboration between researchers, educational designers, and teachers is envisioned to further explore teaching instantaneous speed in grade five.

6.2 The proposed LIT and the aims of the research project

After summarizing the various studies this thesis is comprised of, the focus now shifts towards discussing the findings of the design research project. The proposed LIT for teaching instantaneous speed in 5th grade is discussed in light of the aims put forth in the Introduction of this thesis. This research project was started to answer a call for innovative STEM education in primary school (Gravemeijer, 2009; van Galen & Gravemeijer, 2010; Léna, 2006; Millar & Osborne, 1998) suitable for the information age, which should be inquiry-based, close to students' world view, ICT-rich, and integrated into the primary school curriculum. These characteristics and their implications gave rise to four themes characterizing the proposed LIT and its potential application:

- an innovative approach to teaching instantaneous speed,
- MBL and teachers' professional development,
- computer-enhanced learning in the information society,
- and bridging the gap between students' world-view and STEM.

Before discussing these four themes, it is noted and emphasized that the proposed LIT is not intended as a ready-made product but as a potentially viable theory for other researchers, educational designers, and maybe teachers to apply and adapt to their situation. This aspect of the proposed LIT is intertwined throughout this discussion and will be elaborated further when discussing design research in the next section.

6.2.1 An innovative approach to teaching instantaneous speed

This research can be placed in a long tradition of calculus reform (Tall et al., 2008). In particular, it can be placed among initiatives to teach calculus-like topics in primary school (Nemirovsky, 1993; Thompson, 1994b; Boyd & Rubin, 1996; Nemirovsky et al., 1998; Noble et al., 2001; Stroup, 2002; Ebersbach & Wilkening, 2007; van Galen & Gravemeijer, 2010). According to Tall (2010), calculus reform at large can be characterized by adapting and improving conventional approaches to calculus with the use of ICT (Tall, 2010). Building on Stroup's (2002) qualitative calculus, the proposed LIT deviates significantly from conventional approaches to rate of change in that it tries to support students in developing a non-ratio based understanding of rate. More so, and this makes the proposed LIT truly an innovative approach to learning instantaneous speed, it circumvents the problematic limit concept (Tall, 1993; Tall, 1997; Tall, 2009) all together while still enables students to quantify instantaneous speed.

The proposed LIT tries to support students in expanding their intuitive understanding of instantaneous speed by having them compare constant speed with instantaneous speed in a computer simulation. In the context of filling glassware, the tangibility of the relationship between the constant speed in a highball glass and the instantaneous speed in a cocktail glass allows students to develop a quantitative measure of instantaneous speed that relies on their understanding of constant speed. As a result, average speed takes a less prominent place in the proposed LIT than in conventional approaches to instantaneous speed, which is fortunate because students of this age do have a limited understanding of average speed although it is part of their curriculum.

Moreover, although the proposed LIT is for teaching instantaneous speed in grade five, it might be interesting to explore the usability of its innovative approach to instantaneous speed in higher grade levels as well. A deeper understanding of instantaneous speed might provide students with a strong conceptual basis for coming to understand the limit concept. It might make students more susceptible to the ideas explored in a conventional calculus course.

6.2.2 *Modeling-based learning and teachers' professional development*

The characteristic of inquiry-based learning—students' learning is supported by involving them in real-life situations with an emphasis on questioning, hypothesizing, and experimenting (Léna, 2006; Rocard et al., 2007; J. Osborne & Dillon, 2008)—is covered in the proposed LIT by means of using MBL. MBL is a form of inquiry-based learning built around the idea that modeling is a core activity of science and developing knowledge. From that perspective, MBL seems a natural basis for STEM education. Because it is impossible to know ones mental model, the only way to get any access to the students' mental models is having them express their understanding (Coll et al., 2005; J. Gilbert & Boulter, 1998). An expressed model can be presented, discussed, and evaluated in class, allowing students to refine their mental models, all the while giving the teacher (and researchers) indirect access to their thinking. Although tangible (scale) models seem to be more common in primary education, with "model" is emphatically also meant more abstract models, such as a flowchart, a Cartesian graph, or a simulation (J. Gilbert, 2004; S. Gilbert, 1991).

Key to MBL are suitable classroom social norms that support students to freely express their opinions, ask questions, indicate their doubts or disagreements, and explore alternatives. The teacher plays an important role in creating and maintaining such a supportive learning environment, which is a matter of concern. First of all, Lehrer and Schauble (2010) warns that modeling is not straightforward. It is a skill that students and teachers alike have to develop through modeling. However, because MBL is different from conventional approaches to instruction in primary education, teachers will need support to start with MBL. Secondly, because primary school teachers do not have much expertise teaching STEM and have a poor understanding of STEM (Léna, 2006), modeling as an activity might not be well-understood. In either case, however, we think that the proposed LIT could function as a starting point to introduce primary school teachers to MBL, for example when embedded in a professional development effort.

6.2.3 *Computer-enhanced learning in the information society*

In establishing and maintaining a suitable classroom culture for MBL there is a role for ICT. In particular, flexible and interactive computer simulations enable students to explore phenomena they normally do not have access to (Chang, 2012), therefore enabling students to solve more meaningful, complex, and realistic problems (Ainley et al., 2000). Indeed, although students could explore the context of filling glassware using concrete glasses and water, by offering them a computer simulation of filling glassware, they were able to repeatedly explore filling various glasses without creating a wet mess. The computer simulations offered the students a save environment to explore the situation in detail, formulate and test hypotheses, and discuss, evaluate, and critique their ideas by using the computer simulation in their arguments.

However, what made the computer simulations a truly necessary means of support in the proposed LIT for teaching instantaneous speed was its support for students to construct a dual understanding of graphs and for them to discover and use tools to quantify instantaneous speed. The computer simulation could draw both a line graph and a bar graph. This emphasized and supported the dual understanding of graphs the students were to construct: both as a way to reason about individual data points and a way to reason about the whole continuous process of filling a glass. Moreover, by increasing the number of bars shown, slowly the shape of the curve comes to the fore, offering the students a powerful image to support them in accepting the curve as a good fit to describe their understanding of the situation. It creates an affordance for shuttling back and forth between a continuous and a discrete image of change.

Similarly, once the students construed the highball glass as a measure for instantaneous speed, the highball-glass-tool not only allowed them to put their invention to the test, it also made their theory tangible. The tangent-line-tool allowed students not only to quantitatively explore filling glassware through graphs, but other contexts as well. More interestingly, however, by moving the mouse cursor over the curve while using the tangent-line-tool, the changing speed is aptly visualized, strengthening students' understanding of the relationship between speed and steepness of a curve. Furthermore, this simulation of the changing angle of the tangent line over time offers a potential avenue to support students in developing an understanding of instantaneous speed as a rate. Furthermore, it makes for a tool that could potentially be used to explore all kinds of topics in primary school where change or growth plays a significant role.

It is noted, however, that the use of these computer tools in the proposed LIT is not an end in itself but a means to support students' learning processes while exploring instantaneous speed. In a sense, this use of computer tools is a reflection of how ICT has come to dominate our interactions with dynamic phenomena in our society. As using computer technology has become an integral part of doing STEM, it should also be an integral part of STEM education. As more and more dynamic phenomena are being monitored, and more (real-time) data is becoming available, the potential for meaningful and realistic exploration of these phenomena by students and teachers is enormous. Key to unlocking this potential is realizing that computer technology is an inseparable part of learning and teaching STEM in the information society.

6.2.4 *Bridging the gap between the students' world-view and STEM*

The proposed LIT is close to students' world-view. Obviously, they are very familiar with the context of filling glassware: they have filled glasses and bottles all of their lives. At the same time, although they do have a conception of instantaneous speed in this context from the start, they probably have never thought much of it. This makes filling glassware an ideal context to explore students' covariational reasoning (Swan, 1985; Carlson et al., 2002; McCoy et al., 2012; Gravemeijer, 1984-1988; Castillo-Garsow et al., n.d.; Johnson, 2012; Thompson et al., 2013; Carlson et al., 2002). More so, the context of filling glassware is intrinsically tied to the proposed LIT because it connects constant speed to instantaneous speed in a tangible and visible way that is difficult to realize in the motion context that is commonly used to explore (average) speed in primary school.

On the other hand, due to its simplicity, filling glassware is not a very inspiring context. During the classroom teaching experiments it was observed that some students started getting bored with exploring filling glassware over and over. This context is intended as a starting point for an exploration of a wide range of topics. Any practical adaptation of the proposed LIT should aspire to explore various dynamic phenomena, which binds into the characteristic of integration of new STEM education in the primary school curriculum. There are a number of topics in both the curriculum and current events where rate of change, speed, or growth play a significant role, such as biological growth processes, historical demography, financial crises, physical education, weather, traffic, and so on. In a MBL setting, students can explore these topics in more detail by using graphs and the tangent line tool, which could result in a richer experience. Ultimately, exploring authentic and meaningful situations with tools from STEM might help close the gap between the students' world-view and the reality propagated by the scientific world-view (J. Osborne & Dillon, 2008). After discussing doing design research in the next section, a proposal is made for a possible fruitful way to adapt the proposed LIT that takes into account these characteristics of innovative STEM education.

6.3 *Reflections on design research*

This thesis is as much about doing design research as it is about the development of the proposed LIT for teaching instantaneous speed in grade five. Design research is still an evolving research methodology. Despite the growing body of literature on design research (see for instance Plomp and Nieveen (2013)), there is no text book or manual that delineates how to do a design research project as outlined in Gravemeijer and Cobb (2013). Without any pretension to create a practical manual, this thesis illustrates getting started with design research (Chapter 2), the iterative nature of the design research (Chapter 4), and the place of generating new theory in design research (Chapter 3). In the following, these three characteristics are discussed, followed by the limitations of design research and this design research project in particular.

6.3.1 Doing design research

Design research can be characterized as an iterative process of refining a LIT in multiple design experiments (Gravemeijer & Cobb, 2013). Key to this process is the *initial* LIT. Remember that design research is often used to explore learning that is new somehow, of which not much is known (Kelly, 2013). It is unlikely that a topical literature review will amount to a thorough understanding of the problem at hand: other sources have to be tapped, such as the researcher's prior experiences or a preliminary study. In this respect, design research entails a creative process in which design choices can have long-reaching consequences.

For example, the small-scale study to explore 5th graders' understanding of speed in a situation with two co-varying quantities (Chapter 2) shaped the initial LIT and the design of the learning trajectory in the first design experiment. In the one-on-one teaching experiments of the starting-up phase, the students seemed to react well to the measuring-cup activities, while having more trouble with the graphing activities. Without much thought, the measuring-cup activities were adopted to introduce the students to the context of filling glassware in the first lesson of the learning trajectory in design experiment 1. At the same time, the graphing activities were pushed to the next lessons and were scaffolded by the graphing-component in the computer simulation. Unfortunately, the resulting learning environment promoted discrete thinking, which interfered with students' development of continuous conceptions of speed.

One could wonder if this problem of a discrete learning environment reinforcing discrete thinking could have been prevented. But should it? The design research would have been significantly different: Because discrete thinking was reinforced in the beginning, it became a topic of concern, which directed the retrospective analysis towards inspecting the dichotomy between discrete and continuous images of change and the discovery of the literature on "smooth" and "chunky" thinkers (Castillo-Garsow, 2012; Castillo-Garsow et al., n.d.). Subsequently, discrete and continuous thinking became a central theme in the design research project (Chapter 4). This binds in with the secondary aim of design research to develop theory on more encompassing issues than just the LIT. In this regard, the central theme can be seen as such a more encompassing theoretical issue.

As an aside, it is interesting to note that this dichotomy can also be traced back to the development of the concept of instantaneous rate of change, from a qualitative approach to change in classical antiquity to the quantitative differential calculus approach in the early modern period (Boyer, 1959; Dijksterhuis, 1950). A salient detail in this regard is the fact that Heytesbury's (1335) intuitive notion of instantaneous velocity (Clagett, 1959)—this was the inspiration of using the highball glass' constant speed as a measure for the instantaneous speed in the cocktail glass in the LIT—was formulated in the beginning of this development towards the differential calculus. It might be a fitting start for students to learn about instantaneous speed and offer them a strong conceptual basis for coming to understand the concepts of the differential calculus.

Here the power of iterative development comes to the fore. By refining the LIT in multiple design experiments, this central theme could be tracked across different instructional settings. It adds to the justification of the conjectures generated around the central theme through triangulation. In particular, it contributed to the ecological validity by exploring how similar students in different classrooms struggled to align discrete and continuous images of change. It seems therefore plausible that students in other classrooms will have similar struggles and by elaborating these struggles, potential users of the proposed LIT can recognize these struggles from their own experiences even if they are unfamiliar with teaching instantaneous speed in 5^{th} grade. Smaling (2003) speaks of communicative generalizability in this regard, but beyond that, concerns of generalizability of the findings of the proposed LIT have to be raised. These limitations are discussed in the next section.

Justification of conjectures does not only stem from repeated validation in multiple teaching experiments, but also from the systematical way they are generated during the retrospective analysis by means of abduction. The use of abduction is distinctive to design research and it differentiates design research from experimental research, which builds on induction and deduction. In Chapter 3 it is illuminated how abductive reasoning was applied in the retrospective analysis of the third design experiment to generate the conjecture that students come to the classroom with a continuous conception of speed and only switch to discrete reasoning because of a lack of means for visualizing continuous change. Writing Chapter 3 and explicating the abductive reasoning process was quite enlightening to get a better understanding of how new conjectures are actually generated and why these conjectures are credible. There is a clear relation between an surprising fact that triggers abductive reasoning and the proposed explanation for that fact, the generated conjecture. However, the new claim is justified by it being grounded in the data collection. Here comes a second level of triangulation in play: during the teaching experiment a multitude of data sources is collected, all of which can contribute to support the validation of the newly generated conjecture.

With respect to the iterative nature of design research it is worth noting that the development of the LIT and prototypical instructional sequence is mostly a continuous process. Most conjectures in the LIT are carried over from one design experiment to the next. Similarly, the various computer simulations developed in this thesis share many characteristics. There is a clear line of development from the first computer simulation in the starting-up phase to those in the last design experiment. Doing design research means continuously building on previous experiences, materials, and ideas; design research is emphatically not a sequence of isolated design experiments. It is an iterative and cumulative process

6.3.2 Limitations

Due to the nature of design research there are concerns about the generalizability of the findings of design research. In particular, the classroom teaching experiments took place in gifted (and mixed) classrooms, the instructional sequences that were tried out were very short, and over-all it was tried to set up the learning environment most conducive to gathering data for research. Furthermore, because the proposed LIT is not intended as a ready-made product, there are issues with the application of these findings as the proposed LIT has to be adapted by others to their situation. A potential practical adaptation needs further support. These limitations of design research are discussed in terms of the design research presented in this thesis.

The conjectures of the proposed LIT were generated and validated throughout the design research project based mainly on transcripts of video captured whole-class discussions and collected student products. The extent to which these data faithfully represent the reasoning of *all* students largely depends on the established classroom-culture during the teaching experiments. It appeared that in each design experiment there was a classroom-culture wherein students felt safe to actively participate, freely express themselves, and ask questions. Some students were more strongly present than others, however, who adopted a more cautious attitude towards participation in the whole-class discussions. On the other hand, the aim of this design research was to explore innovative ways of teaching instantaneous speed in 5th grade. This resulted in a potential viable learning trajectory, which is elaborated in the proposed LIT, that delineates a way to support 5^{th} grade students in developing a notion of instantaneous speed that could lead to a more quantitative understanding of instantaneous speed.

At the same time, however, especially because most students participating in the teaching experiments were above average performing students or gifted students, it is likely that teaching instantaneous speed in 5^{th} grade will take more time and effort than suggested by what happened during the teaching experiments. This binds into the issue of generalization of the findings. In the end, the findings are based on the small number of students and classrooms that participated in the teaching experiments. These findings can only be generalized by means of "communicative generalization" (Smaling, 2003), which means that:

> 'it is not so much the researcher but the reader (or potential user) of the research report who figures out to what degree the research results and conclusions are generalizable to people, situations, cases, et cetera, that are relevant to him.' (Smaling, 2003, p. 59)

It is up to the researcher to best support the potential user to transfer the findings, if at all, to their situation (Smaling, 2003). In this sense, the proposed LIT acts as a theory on how instantaneous speed can be taught in 5^{th} grade as a starting point for potential users to build on when they want to teach instantaneous speed early in the mathematics curriculum, develop educational materials for that purpose, or do further research on this topic. As a potential limitation of the transferability of the findings to real-world classrooms, it is noted that to create and maintain a classroom-culture that is conducive to learning instantaneous speed as proposed in the LIT, the teacher plays a key role. However, applying a MBL approach is not straightforward; teachers, in particular those with little affinity with and exposure to STEM, will need further support to successfully apply MBL in their own classrooms.

Furthermore, it is emphasized that parts of the proposed LIT have to be developed further, in particular students' command and understanding of the quantification of constant speed. In this regard, an approach that starts with exploring discrete movements, such as used by Doorman (2005) and van Galen and Gravemeijer (2010), might offer an interesting perspective. Simultaneously, students' experience and skill with graphs has to be developed, for which a reinvention approach (van Galen et al., 2012; diSessa, 1991) seems a fruitful candidate. Finally, during the design experiments, the generalization of students' use of the tangent line has only briefly been explored. In particular, generalization of students' understanding of quantifying instantaneous speed using the tangent line has to be expanded to other contexts as well.

6.4 *Adapting the proposed LIT: A proposal for further research*

Having discussed the proposed LIT and the process of doing design research that led to that proposed LIT, it has become abundantly clear that the proposed LIT is not ready-made. And neither is the instructional sequence from the third design experiment that is instantiated by it. Yes, it is to be qualified as a potentially viable theory for other researchers, educational designers, and teacher to adapt to their situation, but is it actually useful? To answer this question affirmative, a proposal for a viable adaptation is made.

Given the average primary school teacher's limited affinity with and knowledge of (teaching) STEM (Léna, 2006), adapting the proposed LIT by the average teacher on his own is unlikely, and maybe even undesirable. Indeed, because most primary school teachers will not be familiar with calculus (Kaput & Roschelle, 1998) and instantaneous speed is not part of the primary curriculum, it is likely that the average teacher does not possess the necessary understanding of issues related to (teaching) instantaneous speed. This suggests that any adaptation of the proposed LIT requires sufficient and suitable support for the teachers implementing it in their classrooms. This suggests an effort to contribute to teachers' professional development, as was suggested before in subsection 6.2.2.

The proposed LIT has some issues regarding the instructional starting points, in particular the students' lack of graphing experience (Leinhardt et al., 1990) and limited command of average and constant speed. To successfully adapt the proposed LIT means to pay extensive attention to overcome these issues and support students in deepening their understanding of graphs and speed side by side. Furthermore, because this thesis is small in scope, the aim put forward in the Introduction to integrate new STEM education in the primary school curriculum has not been treated at all. This suggests a significantly longer learning trajectory than developed in this thesis.

Taking these issues into account, a long-term design research project is needed that aims at exploring the integration of instantaneous speed in the primary school curriculum. Following the suggestion of Schoenfeld (2009) to perform design research in balanced project teams, ideally researchers, experienced educational designers, and teachers collaborate. The teachers will need (initial) support for learning content-knowledge about instantaneous speed, applying modeling-based learning, including initiating and maintaining suitable classroom social norms, and developing PCK about teaching instantaneous speed in primary school. Gravemeijer and Eerde (2009) speak about "dual design research" in this regard, where the teacher has to learn about (teaching) a topic while supporting students in learning that topic (Gravemeijer & Eerde, 2009). Throughout the design research project, however, they will construct PCK about teaching instantaneous speed, gain expertise with inquiry-based learning approaches and in particular MBL, and deepen their own understanding of instantaneous speed. During this professional development, they become invaluable resources for the project team.

For example, they can help the educational designers in selecting and elaborating suitable topics from the primary school curriculum where change, growth, or speed can play an important role. These topics are ideal candidates for exploration in the long-term learning trajectory on instantaneous speed, taking care of both integrating STEM in the curriculum and exploring topics that are close to the students' world view. Furthermore, as experts on their students' instructional starting points and capabilities, the teachers play an important role in intertwining the learning trajectory on instantaneous speed with the curriculum regarding speed and graphs.

Beyond the practical adaptation of the proposed LIT, in the proposed design research project there is ample room for researchers to study different aspects of learning in real-world classrooms, teaching, and professional development. In doing a long-term design research with a large project team that includes teachers, the promise of design research to bridge the gap between research and practice becomes more credible. Ultimately, the proposed design research project to explore teaching instantaneous speed will result in a better understanding of how to teach instantaneous speed in grade five.

If nothing else, even when instantaneous speed is not taught—it is not part of the primary school curriculum, after all—, the findings presented in this thesis do have practical implications for upper primary education:

1. Delay teaching of average speed. For one thing, because most problem situations that primary school students will encounter at school are linear in nature, average speed does not have much meaning beyond constant speed. Instead, focus on deepening students' qualitative and quantitative understanding of constant speed and ensure they develop a flexible command of different units for constant speed.

2. Support primary school students in developing a good understanding of Cartesian graphs by having them reinvent graphs. The students should be able to express and discuss their understanding of dynamic phenomena by means of representing and interpreting their ideas with graphs.

Summary

Exploring Instantaneous Speed in Grade Five A Design Research

Chapter 1 introduces the topic and set-up of this thesis: In answer to a call for new STEM education in primary school (Léna, 2006; Gravemeijer & Eerde, 2009; Gravemeijer, 2013; van Keulen, 2009; Millar & Osborne, 1998), a design research project was started to answer the question *How to teach instantaneous speed in grade five?* The topic of instantaneous speed was chosen because the interpretation, representation, and manipulation of dynamic phenomena are becoming key activities in the information society and it is firmly rooted in the realm of STEM. Key to a better understanding of dynamic phenomena is the concept of instantaneous speed, which is conventionally taught first in a calculus course in upper secondary education or college. A design research approach was chosen, as design research is well-suited to explore how to teach topics in earlier grade levels than usual (Kelly, 2013).

Design research is an interventionist process of iterative refinement that takes real-world classroom practice into account when creating some instructional artifact and a theory on how that artifact works (Cobb, 2003; The Design-Based Research Collective, 2003; Barab & Squire, 2004; Reimann, 2011; Plomp, 2013; Bakker & van Eerde, 2013). In this thesis, the approach to design research outlined in Gravemeijer and Cobb (2013) was applied to develop a local instruction theory (LIT) on teaching instantaneous speed in grade five. In this approach, a design research project starts by formulating an initial LIT. This LIT is then elaborated, adapted, and refined in multiple design experiments (Cobb, 2003; Gravemeijer & Cobb, 2013). A design experiment consists of three phases: developing an educational design based on the

LIT, trying out and adjusting that design in multiple teaching experiments, and performing a retrospective analysis on the data collected during the teaching experiments to refine the LIT, which is then used in the next design experiment. The retrospective analysis is typically based on the constant comparison approach of Glaser and Strauss (1967). In this project the elaboration of Cobb and Whitenack (1996) was used. In line with the theory-driven nature of design research the focus in this thesis is on the development of the LIT rather than on the development of the educational design. However, instead of giving a continuous narrative detailing this design research, this dissertation focused on four separate studies that emerged from the design research project, which are summarized next. Furthermore, to get a complete overview of the design research project, after discussing the starting-up phase, a short overview of the three design experiments is included as well.

Starting up the design research project

In Chapter 2 it is described how the design research project started by exploring 5th graders' prior conceptions of speed through a literature review that was followed by a preliminary study. In both the literature on primary school students' conceptions of speed and the primary school curriculum, speed is treated as average speed and interpreted as a ratio of distance and time. To overcome students' problems with speed (Groves & Doig, 2003; Thompson, 1994b), Stroup (2002) proposes a "qualitative calculus" approach, which builds on students' qualitative understanding of speed instead of following conventional approaches that favor ratio-based understandings. Because instantaneous characteristics of speed have not been explored in this literature, this review was extended to cover teaching calculus-like topics early in the mathematics curriculum as well. In this research it was found that young students are able to explore speed mathematically using computer simulations and graphs (Nemirovsky, 1993; Thompson, 1994b; Boyd & Rubin, 1996; Nemirovsky et al., 1998; Noble et al., 2001; Stroup, 2002; Ebersbach & Wilkening, 2007; van Galen & Gravemeijer, 2010; Kaput & Schorr, 2007) when these are embedded in a suitable instructional sequence (Gravemeijer et al., 2000), even provided that they lack graphing experience (Leinhardt et al., 1990).

Based on this literature review, a small-scale study was performed to to explore 5th graders' understanding of speed in the context of filling glassware. To that end, a short instructional sequence and a computer simulation were developed and subsequently tried in eight one-on-one teaching experiments (Steffe & Thompson, 2000) in which the students were asked to turn a highball glass, a cocktail glass, and an Erlenmeyer flask into a measuring cup, draw a graph of filling the glass, and evaluate their work using the computer simulation. The data collected during these experiments were analyzed using Carlson et. al.'s covariation framework (Carlson et al., 2002): Students' utterances were coded for one of five developmental levels of covariational reasoning, which ranged from a basic understanding of co-varying quantities to an understanding of a dynamic situation in terms of instantaneous speed. It was found that the students were quite capable in estimating the relative amount of change at certain points or intervals, but they almost never talked about speed. It was also concluded that, due to the students' limited vocabulary and graphing skills, the covariation framework did not offer the clarity on instantaneous aspects of change to be a good instrument to study 5th grade students' conceptions of instantaneous speed.

Nevertheless, the findings of the literature review and the experiences during the teaching experiments allowed for the formulation of an initial LIT that aimed at expanding on Stroup (2002)'s qualitative calculus approach by supporting 5th graders in developing both a qualitative and a quantitative understanding of speed. After exploring the relation between a cocktail glass' shape and the rising speed, students may realize that the instantaneous rising speed in the cocktail glass at a given height is equal to the constant rising speed in an highball glass that has the same width. This would eventually enable students to quantify the instantaneous rising speed in a point by computing the constant rising speed of the corresponding highball glass. Next, building on the correspondence between a highball glass's graph and the steepness in a point of a curve, the students might construe the tangent line in a point of a curve as an indicator for the instantaneous speed. Given students' familiarity with constant speed and graphing linear situations, students then could quantify the instantaneous speed by computing the rise over run of the tangent line.

Overview of the three design experiments

After the starting-up phase of the design research project, three design experiments were performed. The procedure of analysis used during each of the three design experiments was a two-step retrospective analysis modeled after Glaser and Strauss's (1967) comparative method. In particular, the elaboration of Cobb and Whitenack (1996) on this method was used. After formulating conjectures about *What happened?* during the teaching experiments and testing these conjectures against the data collected, a second round of analysis was carried out by formulating conjectures about *Why did it happen?*. These conjectures were also tested against the data collection. The data was collected during the following three design experiments:

1. The first design experiment revolved around an instructional sequence that started with students making measuring cups from various glasses, followed by a lesson on exploring quantification of speed in the highball glass. Only then, in the third lesson, the attention shifted to instantaneous speed. Following the initial LIT, it was expected that students would come to see that they could determine the instantaneous speed at a specific point in the cocktail glass by computing the constant speed of a computer-drawn virtual highball glass with the same width. By directly linking the virtual highball glass with its graph as a tangent line on the cocktail glass' curve, the students were expected to come to accept the tangent line as a tool to measure instantaneous speed in a graph. Although the students could handle the tangent-line-tool, their level of understanding was doubted. Furthermore, they did not manage to discover a continuous graph; it was conjectured that starting with the linear situation of the highball glass might have put them on the wrong track.
2. In the small-scale design experiment that followed, it was decided to remove the measuring cup activities and have the students' learning process revolve around the non-linear situation of filling the cocktail glass. To increase conceptual discussion about speed a modeling-based learning (MBL) approach was selected, which helped to make the students' thinking visible and a topic of discussion. During the teaching experiments the students were repeatedly asked to model filling a cocktail glass, and to improve their models after exploring the situation in a computer simulation. It was expected that, once the Cartesian graph was introduced by the teacher, they would be able to use it to predict the water level height at any moment and to connect its shape to their image of filling glassware. The students were expected to come to see that the cocktail glass' curve is as steep as the highball glass' straight line in the point they have the same width. During the teaching experiments, the students did construe the tangent line to the cocktail glass' curve parallel to the graph of the highball glass as an indicator of the speed in a given point. However, there was no success with developing continuous graphs, which seemed to suggest that maybe students of this age are chunky thinkers (Castillo-Garsow, 2012). Furthermore, the students' lack of understanding of, and fluency with, measures of speed revealed itself.
3. Design Experiment 3 was a turning point. The students in one of the two classes that the experiment was tried in, showed that they were able to invent a continuous graph by themselves. The catalyst proved to be a critical reflection on the shape of the segmented-line graph, while using their understanding of

the relationship between a glass' width and speed. This meant that they were not chunky thinkers; they were continuous thinkers (Castillo-Garsow, 2012) who have difficulty with graphing. In addition, it showed that there is another road to the continuous graph, which was followed by the students in the other classroom where the students came to understand the continuous graph via shrinking the intervals of a bar chart. Furthermore, the students could build on those ideas to come to grips with the shape of the graph of a cooling process by assuming that the tangent line in a given point depends on the difference between the actual temperature and the final temperature. However, even though the students had a tool to measure instantaneous speeds, they did not yet develop a sound understanding of how to quantify speed. They appeared to be hampered by the fact that they did not have a sound basis for calculating speeds.

The generation of new explanatory conjectures in design research

Chapter 3 focuses on the generation of new theory during the retrospective analysis, highlighting the role of abductive reasoning (Fann, 1970)—which can be characterized as noticing that a certain observation needs an explanation and seeking to find the simplest and most likely explanation of that observation. It is shown how abductive reasoning led to the generation of the conjecture during the retrospective analysis of the third design experiment that primary school students come to the classroom with a continuous conception of speed and only switch to discrete reasoning because of a lack of means for visualizing continuous change. This, in turn, led to the realization that average rate of change is a hindrance rather than a necessity in teaching instantaneous rate of change in primary school. It is illuminated how abduction plays a specific role in design research by describing both the teaching experiment leading up to the unexpected event and the two-step retrospective analysis that followed.

In design experiment 3, a four-lesson instructional sequence was developed and tested in two gifted 4^{th}-6^{th} grade classrooms (C1 and C2) taught by the same teacher. In the first lesson, the students modeled filling a cocktail glass four times, during which the discrete snapshots model became taken-as-shared in both classrooms. At the start of the second lesson the students were asked to create a minimalist model, most of which were discrete representations. However, in each classroom, there was one graph-like model with continuous characteristics. In C1, while discussing the student-drawn graph-like minimalist model, the students argued that the straight line segment had to be a curve: they invented the curve by themselves. In C2, the graph-like model was not discussed and the students' learning trajectory followed the one anticipated in the LIT. This unexpected difference led to a process of abductive reasoning during the retrospective analysis.

The retrospective analysis was based on a two-step method based on Glaser and Strauss's (1967) comparative method: after testing conjectures about what happened, conjectures about why that did happen were formulated and tested against the data collected during the teaching experiments. It showed that the students' reasoning was grounded in continuous reasoning, while discrete reasoning functioned as a tool to get a handle on continuous processes. Furthermore, the students easily reasoned about constantly changing speed, which implies a conception of instantaneous speed. This observation triggered a process of abduction at the design research level, which led to the conjecture that starting with average speed is problematic: it promotes discrete thinking and could be the source of problematic chunky thinkers (Castillo-Garsow, 2012). It would make more sense to explore constant speed, which subsequently can be connected to the students' notion of instantaneous speed.

However, the generalization of these findings were possibly limited by the uniqueness of the classroom situation. Furthermore, abduction does not offer the same rigor as deduction and induction. However, the primary goal of design research is to find out how things work, not to establish for a fact how things are. By being explicit about the abductive argument underlying the development of the LIT, special attention is paid to the justifications common to design research, such as ecological validity, trackability (Smaling, 1990), process oriented causality (Maxwell, 2004) and consilience (Gould, 2011).

Design research as an augmented form of educational design

Design research builds on educational design to develop both a product and a theory detailing how and why that product works. To develop theory implies a commitment to strengthen the credibility of the theory by substantiating its claims and a commitment to allow other researchers to assess the trustworthiness of the process leading up to those claims. The latter can be satisfied by enabling outsiders to retrace the process by which those claims are produced (Smaling, 1990), which means to give a detailed account of the design research process and the researcher's own learning process embedded in it. In Chapter 4, it is argued that this method of reporting on the learning process of the researchers can function as a paradigm for the way educational designers might want to document their practices and knowledge. To offer an example, the researcher's own learning process is elaborated by tracking the development of the instructional sequence from the starting-up phase through the three subsequent design experiments in terms of a framework encompassing both the design decisions and the rationale for those decisions. Such a framework of reference may take the form of a LIT that offers a rationale for the instructional sequence that is developed alongside the LIT. In this manner, design research offers a different kind of support for teachers than most textbooks do.

Apart from offering an example of documenting instructional design decisions and practices, Chapter 4 also elaborates on what makes design research credible, even though it does not follow the classical research method of an (quasi-)experimental design. The theoretical findings can be substantiated by the virtual repeatability or trackability of one's research by other researchers (Smaling, 1990). The goal of design research is to generate a theory on how the intervention works. To develop theory, two methods are used in design research: validating existing conjectures and generating new explanatory conjectures. Conjectures that are confirmed by the students' actual learning process remain part of the LIT and are tried and refined again in the next design experiment, offering a form of triangulation that adds to the understanding of students' learning processes in terms of these conjectures. New explanatory conjectures are generated through abductive reasoning, but only claims about the students who participated in the experiments can be made. They have to be grounded carefully in the observational data to make sure that the conclusions are valid for the majority of the students in the teaching experiments.

The goal is to come to understand the specific characteristics of the investigated learning ecology in order to develop theoretical tools that make it possible to come to grips with the same phenomenon in other learning ecologies. The LIT offers a theory of how the intervention works, which teachers and instructional designers can adjust and adapt. Design research can be taken as a paradigm that may show educational designers the value of documenting design decisions and anticipated learning processes. In this way, design research can be seen as an augmented form of educational design, which offers educational designers indications on how to handle the issue of documenting their practices and knowledge.

A proposed LIT on teaching instantaneous speed in 5th grade

By capitalizing on the results of the various design experiments a LIT on teaching instantaneous speed in 5th grade is proposed in Chapter 5. This chapter starts by summarizing the theoretical background and detailing the methodology used in this research project, design research. More specifically, the theoretical underpinnings of the LIT are elaborated in terms of the three instructional design heuristics of Realistic Mathematics Education: guided (re)invention, didactical phenomenology, and emergent modeling. The proposed LIT is based on the patterns in students' learning processes that were identified in the data of the various design experiments. This allowed for a triangulation on two levels. At the level of a single design experiment a multitude of data was collected and used to validate and generate conjectures, and these conjectures were validated or refuted in multiple design experiments.

Reporting on the results, basically Cobb et. al.'s (in press) recommendation for an argumentative grammar for design experiments was followed, which requires the justification of the theoretical findings of a design experiment by a) showing that the students' learning process is due to their participation in the design experiment, b) describing that learning process, and c) enumerating the necessary means of support for that learning process to occur. The first requirement is self-evident as 5th graders are not taught on this topic. The other two requirements are split into a documentation of the key learning processes, and a separate description of the envisioned learning process and the means of support that learning process.

1. The patterns that emerged with respect to students' key learning moments are put in context by a description of students' instructional starting points. These starting points were the same in each teaching experiment: the students had limited graphing experience, they had trouble computing speeds, and they were thinking in terms of instantaneous speed from the start. There is no indication that the situation will be much different in other classrooms. Given these starting points, several key learning moments were identified in the data. Notably, the students were familiar with linearity, but broke through the linearity illusion (de Bock et al., 2002) easily when seeing the cocktail glass fill up. They understood the relationship between a glass width and its speed, which allowed them to realize when the speed in the cocktail glass and highball glass would be the same. Despite the students' limited graphing experience, once the curve was introduced—in one classroom the students even invented

it themselves—they accepted it as a better model than the discrete snapshots models they had created earlier. They were able to construe the tangent line as an indicator of the speed in a given point by combining the cocktail glass' graph and the graph of a highball glass.

2. Based on these key learning moments, the proposed LIT is formulated in Chapter 5 in terms of a postulated students' learning processes and the potential means of support necessary for those learning processes to emerge. The students' potential learning processes may be summarized as follows:

   Given a cocktail glass, students are given the task of making a drawing of how the water height changes when the glass fills up. After observing the glass fill up, they notice that the water level rises slower and slower, and they realize this is the result of the glass' increasing width. This realization allows the students to form valid expectations about the process of filling glassware and they come to depict it both as a discrete bar chart as well as a continuous graph. It is expected that the students link the curve of the continuous graph with the continuous change of the speed of the rising water— at every moment that speed is different—based on their intuitive notion of instantaneous speed.

   This conception is deepened both qualitatively and quantitatively by exploring two avenues of thought. First, by comparing the speed in the cocktail glass with the constant speed in a cylindrical highball glass and answering the question when the water rises with the same speed in both glasses. Then the constant speed of an imaginary highball glass can become a measure for the instantaneous speed in the cocktail glass. Second, building on that understanding, trying to measure speed in a graph by interpreting the straight line graph of the highball glass as a tangent line on the curve of the cocktail glass. Throughout this process, the representations of the speed in the highball glass act as an emergent model: the model of water height become a model for speed. Finally, students' understanding of speed in terms of graphs and tangent-line can be translated to other contexts as well.

   For this learning process to emerge, three potential necessary means of support are identified: the context of filling glassware, computer simulations combined with graphs, and the modeling-based learning (MBL) approach to learning. The context of filling glassware visually connects constant speed, which students already know, to instantaneous speed; picturing the highball glass with the cocktail glass offers a very powerful image enabling students to invent the highball glass as a tool to measure instantaneous speed. Combining interactive simulations with graphs enabled an inquiry-based learning approach where students and teacher could safely explore speed and graphing. Furthermore, offering both discrete and continuous representations allowed for an affordance for shuttling back and forth between continuous and discrete reasoning. By using an MBL approach the class-based discussions about speed became more conceptual in nature while giving the teacher indirect access to the students' mental models. This allowed the teacher to better support students in constructing a deeper understanding of graphs in relation to speed.

Unique to the presented approach to teaching instantaneous speed is that it circumvents the troublesome limit concept while supporting students to come to understand and quantify instantaneous speed. This means that compared to conventional approaches, average speed plays a minimal role.

Although the proposed LIT is considered a potential viable theory, it clearly cannot easily be used in regular classrooms due to issues of teacher professional development and the fact that the corresponding instructional sequence has to be elaborated to cater for the students' limited graphing abilities and their limited understanding of quantifying constant speed. For others to adapt the LIT to their own situation, the proposed LIT should be transferable (Smaling, 2003). This means that the findings should both be plausible for others and that they should be able to ascertain the potential applicability to their situation. Some aspects of the proposed LIT are problematic in this regard, implying potential avenues for further development and research.

Discussion

In Chapter 6, the findings of the design research project are discussed in light of the aims put forth in the Introduction, by reflecting on doing design research, and by proposing a viable adaptation of the LIT.

Characterizing the proposed LIT

The proposed LIT and its potential application can be characterized by four themes:

1. Building on Stroup's (2002) qualitative calculus, the proposed LIT deviates significantly from conventional approaches to rate of change in that it tries to support students in developing a non-ratio based understanding of rate and

by circumventing the problematic limit concept (Tall, 1993; Tall, 2009), which makes it a truly innovative approach to learning instantaneous speed.

2. MBL is a form of inquiry-based learning built around the idea that modeling is a core activity of science and seems a natural basis for STEM education. Key to MBL are suitable classroom social norms that support students to freely express their opinions, ask questions, indicate their doubts or disagreements, and explore alternatives. The teacher plays an important role in creating and maintaining such a supportive learning environment, which is a matter of concern: teachers will need support to start with MBL, but as primary school teachers do not have much expertise teaching STEM and have a poor understanding of STEM (Léna, 2006), modeling as an activity might not be well-understood.
3. In establishing and maintaining a suitable classroom culture for MBL there is also a role for ICT: flexible and interactive computer simulations enable students to explore phenomena they normally do not have access to (Chang, 2012), therefore enabling students to solve more meaningful, complex, and realistic problems (Ainley et al., 2000). During the teaching experiments, the computer simulations offered the students a save environment to explore the situation in detail, formulate and test hypotheses, and discuss, evaluate, and critique their ideas by using the computer simulation in their arguments. However, what made the computer simulations a truly necessary means of support in the proposed LIT for teaching instantaneous speed was its support for students to construct an understanding of graphs and for them to discover and use tools to quantify instantaneous speed; it creates an affordance for shuttling back and forth between a continuous and a discrete image of change.
4. The context of filling glassware is intrinsically tied to the proposed LIT because it connects constant speed to instantaneous speed in a tangible and visible way that is difficult to realize in the motion context that is commonly used to explore (average) speed in primary school. On the other hand, due to its simplicity, filling glassware is not a very inspiring context. During the classroom teaching experiments it was observed that some students started getting bored with exploring filling glassware over and over. This context is intended as a starting point for an exploration of a wide range of topics. Any practical adaptation of the proposed LIT should aspire to explore various dynamic phenomena, which binds into the characteristic of integration of new STEM education in the primary school curriculum.

Reflections on design research

This thesis is as much about doing design research as it is about the development of the proposed LIT. Design research is still an evolving research methodology. Despite the growing body of literature on design research (see for instance Plomp and Nieveen (2013) and Prediger, Gravemeijer, and Confrey (2015)), there is no text book or manual that delineates how to do a design research project as outlined in Gravemeijer and Cobb (2013). This thesis illustrates getting started with design research (Chapter 2), the iterative nature of the design research (Chapter 4), and the place of generating new theory in design research (Chapter 3).

However, due to the nature of design research there are concerns about the generalizability of the findings. Moreover, the classroom teaching experiments took place in gifted (and mixed) classrooms, the instructional sequences that were tried out were very short, and over-all it was tried to set up the learning environment most conducive to gathering data for research. The conjectures of the proposed LIT were generated and validated throughout the design research project based mainly on transcripts of video captured whole-class discussions and collected student products. Some students were more strongly present than others, which raises the question to which extend the data faithfully represents the reasoning of all students. Furthermore, the findings can only be generalized by means of "communicative generalization" (Smaling, 2003), which means that it is up to the researcher to best support the potential user to transfer the findings, if at all, to their situation (Smaling, 2003). In this sense, the proposed LIT acts as a theory on how instantaneous speed can be taught in 5^{th} grade as a starting point for potential users to build on.

Adapting the proposed LIT: A proposal for further research

The proposed LIT is not ready-made. And neither is the instructional sequence from the third design experiment that is instantiated by it. However, a long-term design research project is proposed that aims at exploring the integration of instantaneous speed in the primary school curriculum, ideally with collaboration of researchers, experienced educational designers, and teachers. The teachers will need (initial) support for learning content-knowledge about instantaneous speed, applying modeling-based learning, including establishing and maintaining suitable classroom social norms, and developing PCK about teaching instantaneous speed in primary school. During this professional development, they become invaluable resources for the project team to help the educational designers in selecting and elaborating suitable topics from the primary school curriculum where change, growth, or speed can play an important role. These topics might be ideal candidates for exploration in the long-term learning trajectory on instantaneous speed, taking care of both integrating STEM in the curriculum and exploring topics that are close to the students' world view. Furthermore, as experts on their students' instructional starting points and capabilities, the teachers play an important role in intertwining the learning trajectory on instantaneous speed with the curriculum regarding speed and graphs.

Beyond the practical adaptation of the proposed LIT, in such a design research project there would be ample room for researchers to study different aspects of learning in real-world classrooms, teaching, and professional development. In doing a long-term design research with a large project team that includes teachers, the promise of design research to bridge the gap between research and practice becomes more credible. Ultimately, the proposed design research project to explore teaching instantaneous speed will result in a better understanding of how to teach instantaneous speed in grade five. Even when instantaneous speed is not taught in primary school, it is not part of the curriculum after all, the findings presented in this thesis do have practical implications for upper primary education as well. Delay teaching of average speed, but focus on deepening students' qualitative and quantitative understanding of constant speed and ensure they develop a flexible command of different units for constant speed. And support primary school students in developing a good understanding of Cartesian graphs by having them reinvent graphs to allow them to express and discuss their understanding of dynamic phenomena through graphs.

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Curriculum vitae

Huub de Beer was born on 31-05-1981 in Reusel, the Netherlands. After graduating high school (academic track; VWO) in 2000 at the Pius X College in Bladel, he studied Computer Science and Engineering at the Eindhoven University of Technology (TU/e), the Netherlands. In 2006 he graduated this 5-year engineering program *cum laude* with a thesis on *The History of the ALGOL Effort*. Meanwhile, from 2001 onward, he also studied History at the Utrecht University, the Netherlands, finishing his bachelors degree in 2006 as well.

In 2007, Huub started working as a historian at the University of Amsterdam on a project aiming at detailing the Dutch computer pioneering period (1945-1965). After a year he made a career-switch towards teaching computer science and enrolled in the Masters program *Science Education and Communication* at the Eindhoven School of Education at the TU/e. He graduated a year later on *Pedagogical Content Knowledge of Teachers Teaching an Introductory Programming Course* while working as a teacher at Pleincollege Eckart in Eindhoven. For three years he worked as a teacher, teaching computer science and ICT in grades 10-12 and 9 respectively.

Meanwhile, in February, 2010 he started a PhD project at the TU/e. The results of this project are presented in this dissertation. Since 2015 he is employed as a programmer.

List of Publications

submitted

- Beer, H. de, Gravemeijer, K., & Eijck, M. van. (submitted) Investigating 5th grade students' level of covariational reasoning.
- Beer, H. de, Gravemeijer, K., & Eijck, M. van. (submitted) Design research as an augmented form of educational design: Teaching instantaneous speed in fifth grade.
- Beer, H. de, Gravemeijer, K., & Eijck, M. van. (submitted) A Proposed Local Instruction Theory for Teaching Instantaneous Speed in Grade Five.

2015

- de Beer, Huub, Koeno Gravemeijer, & Michiel van Eijck (2015) 'Discrete and continuous reasoning about change in primary school classrooms' *ZDM Mathematics Education*, doi:10.1007/s11858-015-0684-5

2013

- de Beer, Huub, Michiel van Eijck, & Koeno Gravemeijer (2013) 'Differentiaalrekening op de basisschool', paper presented at the ORD 2013, Brussel.

2011

- de Beer, Huub, Michiel van Eijck, & Koeno Gravemeijer (2011) 'Design Principles for Teaching Primary Calculus', paper presented at the 2011 Joint Winter School, Hamburg.

- de Beer, Huub, Michiel van Eijck, & Koeno Gravemeijer (2011) 'On Teaching Primary Calculus: First Step in a Design Research and some Issues', paper for round table session at the ICO Fall School 2011, Eindhoven.

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- de Beer, Huub, Michiel van Eijck, & Koeno Gravemeijer (2010) 'Towards Teaching the Concept of Compound Variable Quantities in Primary Education', poster presented at the 2010 Joint Winter School, Munich.

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