Design Principles for Teaching Primary Calculus

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ICT enables teaching calculus already in primary school. The aim of this study is to find design principles for an instructional sequence on primary calculus for 5th grade. Design principles are found in the literature on teaching calculus early and by determining 5th graders' previous knowledge of change. Common characteristics of initiatives to teach calculus early are the use of ICT and graphs, a preference for average rate of change, and a preference for the motion problem domain. Traditionally, calculus education is based on the standard mathematical definition of instantaneous rate of change. That implies teaching average rate of change first and using the conceptually difficult limit concept, making it unsuitable for primary education. To truly understand change is to understand it in different domains, an unilateral focus on the motion domain might be detrimental for learning about change. In our study to determine 5th graders' current level of reasoning about change we found their reasoning to be limited to thinking about change in terms of the amount of change. Furthermore the participants needed to break through the linearity illusion before they were able to tackle non-linear problems of change.

1 Introduction

The aim of this study is to find design principles for an inquiry-based and ICT-rich instructional sequence on primary calculus for 5th grade that can be integrated in the curriculum and is close to the world-view of fifth graders. To better understand how to teach calculus already in 5th grade we intend to do a design research (Gravemeijer & Cobb, 2006) to develop a local instruction theory and an instructional sequence on primary calculus. Design principles guide the development of both theory and educational artifact.

Teaching calculus early by using ICT is not a new idea. Most initiatives on teaching calculus early share some common characteristics: almost all use ICT and graphs to enable younger students to work on problems of change. With "teaching calculus early" we denote all initiatives that teach calculus-related concepts before calculus is traditionally taught. In Section 3 the literature on teaching calculus early is discussed to distill design principles.

Not much is known about what 5th graders already do know about change. Students' previous knowledge indicates the starting points of an instructional sequence and are another source of design principles. To get a reasonable indication of 5th graders' current level of reasoning about change a study is performed (see Section 4). But first the question is answered: Why teaching primary calculus?

2 Why teaching primary calculus?

The world around us changes continuously; change is everywhere. In the scientific world-view the concept of change became to mean a relation of a property of an object in consecutive points in time (Mortensen, 2008). This relation can be described mathematically by a continuous function denoting the total amount of change. The differential of this function describes the instantaneous rate of change.

The reversibility between instantaneous rate of change and total amount of change is a key concept of the calculus.

Traditionally calculus is taught in secondary education and up. As students learning calculus already should have a good understanding of functions and algebra, — taught in the early years of secondary education —, traditional calculus is unsuitable for primary education. However, ICT enables primary students to reason about change without the need to know these advanced mathematical concepts. Walter Stroup coined the term “qualitative calculus” to denote this kind of calculus (Stroup, 2002). It is qualitative in the sense that students understand change in terms of “how fast” and “how much” and in terms of the shape of a graph and its slope. We want students also to apply their understanding of change in realistic problem situations, so they need to be able to quantify change as well. As we focus on teaching calculus-like concepts to primary school children in particular, we introduce the term “primary calculus”.

3 Literature on teaching calculus early

3.1 ICT and graphs enable teaching calculus early

When computers became more affordable, computers entered education at a larger scale (Molnar, 1997; Plomp & Pelgrum, 1991; Pelgrum & Plomp, 1993). That gave a new impulse to educational research on teaching and learning about change (Stroup, 2002). Since the late 1980s there have been a number of research projects on teaching calculus-related concepts in middle school or primary school using then modern computer technology. Some projects used flexible programming environments, like LOGO (Papert, 1993) and Boxer (diSessa, Hammer, Sherin, & Kolpakowski, 1991; DiSessa, 2001). Originally, these flexible programmable environments were called “microworlds”, but over time the term “microworld’ has become to include all kinds of computational environments (Edwards, 1998; Hoyles, Noss, & Adamson, 2002). In most research projects on teaching calculus early, however, less flexible microworlds were used because using software in education adds an overhead to learning (Hoyles & Noss, 2003).

Microworlds that simulate a situation of change are based on a mathematical model. The computer simulation encapsulates the mathematical model and hides advanced mathematical concepts from the students. The students can explore the situation of change via this easy accessible representation. The basic premise of teaching primary calculus is that no advanced mathematical knowledge is needed to use a microworld to study change.

Another common feature among these initiatives is the use of graphs to represent change. Graphs are the main communication method of calculus (Boyd & Rubin, 1996; van Galen & Gravemeijer, 2010; Doorman & Gravemeijer, 2009) and connected to differentiation and integration. Graphs enable primary students to start reasoning about qualitative and quantitative aspects of change. In a qualitative sense, a student is able to interpret how fast and in what direction some quantity is changing. Graphs enable students to quantify change and to compare situations of change in more detail. ICT enables dynamic representations, linking multiple representations, and supplantation of an external representation of operations and linking it to a graph (Vogel, Girwidz, & Engel, 2007). ICT enables younger students to work with graphs more easily and to solve more complex problems (Ainley, Pratt, & Nardi, 2001).

Using graphs is not straightforward. Problems of students regarding the interpretation and the creation of graphs are reported in the literature (Leinhardt, Zaslavsky, & Stein, 1990; Ainley et al., 2001; Barton, 1997; Lowrie & Diezmann, 2007; Vogel et al., 2007), ranging from interpreting graphs as pictures, confusing slope and height, confusing intervals with points, to having difficulty connecting abstract graphs with concrete situations. Although similar problems are encountered by younger students (Ainley et al., 2001; Garcia Garcia & Cox, n.d.; Garcia & Cox, 2010), there are indications that younger students are quite capable using graphs (Phillips, 1997; van den Berg, Schweickert, & Manneveld, 2009), especially given their lack of graphing experience.

Using ICT and graphs enable younger students to reason about change without interference of thinking in terms of algebraic formulas and procedures. In the traditional calculus curriculum algebra is inextricably connected to reasoning about change. Teaching primary calculus focuses on the conceptual problems of reasoning about change instead of focusing also on algebraic procedures and advanced mathematical concepts.

_________________________ *Design Principles* ____

- Encapsulate the mathematical model of a situation of change in a suitable microworld
- Graphs are the communication method of calculus
- Linking of multiple dynamic representations and supplantation support younger students to tackle more complex problems
- A suitable microworld can enables students to focus on conceptual problems instead of mathematical procedures

3.2 Focus on average rate of change and an alternative definition of instantaneous rate of change

In the literature on teaching calculus early there is a preference to teach average rate of change first. One explanation is the prevalence of the standard mathematical definition of instantaneous rate of change. This definition is based on the limit of average rate of change on an interval, as the length of that interval approaches zero.[1] The average rate of change on that tiny interval approaches the instantaneous rate of change at a moment. The idea of limit is difficult to understand (Cornu, 1991) and is one of the prominent problems students have to overcome when learning traditional calculus (Tall, 1993); it is unsuitable for primary education.

Although the traditional mathematical definition of instantaneous rate of change is ubiquitous, alternative definitions can be found in the history of mathematics. Until the fourteenth century, change was seen as a quality and represented mathematically by constants. Then scholastics started quantifying variation (Boyer, 1959). William Heytesbury defined his intuitive notion of instantaneous velocity as: ‘a nonuniform or instantaneous velocity (...) [is measured] by the distance which *would* be traversed by such a point, *if* it were moved uniformly over such or such a period of time at that degree of velocity with which it is moved in that assigned instant.’ (from Heytesbury’s *Regule solvendi sophismata* (1335), as cited in Clagett, 1959, pp. 235—237) This intuitive notion did exist long before today’s standard mathematical definition of instantaneous rate of change, suggesting it might be more compatible with our own intuitive conceptions of change and is more suitable for primary education.

Another explanation for focusing on average rate of change first is a direct consequence of a preference for linear problem situations in education (de Bock, van Dooren, Janssens, & Verschaffel, 2002; Van Dooren, Ebersbach, & Verschaffel, 2010). In a linear situation the instantaneous rate of change equals average rate of change[2] and does not have any meaning beyond average rate of change. Having two different concepts that seem to be the same is confusing. In many of the research projects on teaching calculus early, the chosen problem situations were linear in nature. As younger students are quite capable to tackle more complex problems (Sabelli, 2006), the “linearity illusion” (de Bock et al., 2002) can be broken by using non-linear problem situations.

_________________________ *Design Principles* ____

- Use a more intuitive (Heytesbury’s) notion of instantaneous velocity as basis for understanding instantaneous rate of change
- Use non-linear problem situations

3.3 The problematic motion domain

Another common theme among initiatives of teaching calculus early is the predominance of the motion domain. Motion is essential to our being; from birth onward we move and perceive motion.

Kaput and Roschelle (1998) prefer the motion domain for teaching calculus early because the motion domain ‘ties the mathematics of change to its historical and familiar roots in experienced motion.’ (Kaput & Roschelle, 1998, p. 163) Many other researchers, however, give no reasons why they selected the motion domain, suggesting it to be the natural problem domain for teaching about change.

Students do have a lot of intuitive knowledge about motion (Wilkening & Huber, 2002), enabling them to solve a whole range of motion-related problems. However, to engage students to think beyond their intuitive notions might be difficult in the motion domain. When students can solve problems with their existing intuitive knowledge there is not much incentive to adopt a more formal concept of change. Furthermore, especially because students are well-versed in the motion domain, the notion of change might be implicit in cases of motion and therefore less transferable to other domains of change. There have been a small number of research projects on teaching calculus early that did also use different problem domains besides motion indicating that there is more to learning about change than motion.

Design Principles

- Do not use primarily the motion domain to teach a more formal concept of change
- To understand change is to understand change in different domains

4 5th graders’ level of reasoning about change: a study

4.1 Overview of the study

Not much is known about what primary students already do know about change. To be able to develop an instructional sequence to teach primary calculus and a theory on how to teach that sequence, we need to know what 5th graders already do know about change.

Carlson, Jacobs, Coe, Larsen, and Hsu (2002) created a framework to determine and analyze students’ “covariational reasoning” (Carlson, Jacobs, et al., 2002; Carlson, Oehrtman, & Engelke, 2010). They define covariational reasoning as: ‘the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other.’(Carlson, Jacobs, et al., 2002, p. 354) They use their framework to analyze college students’ understanding of situations of change. Their framework consists of five mental actions. Students’ mental actions appear as their verbal and representational utterances. As such, the mental actions are only indirectly discernible. On top of these five mental actions, five developmental levels are distinguished. A student reasons on a certain level (> *2*) if she performs the mental action corresponding to that level and mental actions of the first two levels. However, Carlson, Jacobs, et al. (2002) warn for so called pseudo-behavior (Vinner, 1997) as students can display behavior of a certain level without being on that level of reasoning.

Primary students do not know algebra, functions, and calculus as do college students the framework is based on. Students use these mathematical concepts to build and interpret mathematical models of situations of change. A mathematical model can also be made accessible through a microworld, enabling primary students to study the underlying mathematical model and reason about change. The covariation framework is adapted to determine and analyze the level of reasoning about two co-varying quantities of primary students by using a suitable microworld.

Based on this covariation framework an one-on-one teaching experiment (Steffe & Thompson, 2000) and a suitable microworld were developed to determine what conceptual problems 5th graders do encounter while reasoning about a situation of change by answering the following research questions:

1. At what level in the covariation framework do 5th graders reason about change?
2. How does the teaching experiment and microworld help to come to this level of reasoning?
3. How develops their reasoning about change during this teaching experiment?

4.2 The actual experiments

Eight teaching experiments were carried out on three separate days. The three experiments on the first day were part of a pilot to test both teaching experiment and microworld. In these experiments four 5th graders took part; the third experiment was performed with two instead of one student. During the second and third day of experiments, respectively three and two experiments were carried out with one participant each. The participants were average to above average performing 5th graders. After the pilot and the second day of experiments both experiment and microworld were adapted to better capture the participants' reasoning about change.

In every experiment the researcher and the participant's teacher took part. The researcher gave the initial instructions. During the experiment both teacher and researcher prodded the participant to explain his or her reasoning. Each experiment took between 25 and 45 minutes. All objects in the microworld, except the water tap, were also available to the student as real-life objects on the table. The whole experiment was videotaped to capture verbal and gestural utterances of the participant. To capture the use of the microworld, the computer session was captured using screen capture software.

4.3 Development of experiment and microworld

The participants were asked to solve problems about filling glasses with water. Primary students are quite familiar with filling glasses and this non-motion domain allows for non-linear problem situations of change. The experiments are divided into three parts: an introduction, the core, and an informal evaluation of the experiment at the end. In the introduction the participant is informed about the goals and set-up of the experiment followed by a couple of questions about a measuring cup to initiate thinking in terms of volume and water level. The core of the experiment consist of three increasingly more difficult problems: filling a long drink glass, filling a cocktail glass, and filling an Erlenmeyer flask. The first problem is linear to make it a suitable problem to introduce the participants to the FlaskFiller microworld[3] and the problem domain. The other two problems are non-linear.

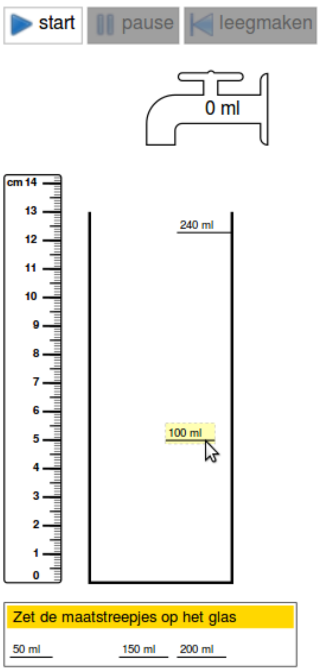

Figure 1: Creating a measuring cup of a long drink glass in the FlaskFiller microworld: drag the given measure lines to the correct position on the glass.

In the pilot all three problems had the same tasks. First, the participants created a measuring cup of the glass by dragging given measure lines to the correct position on the glass (see Figure 1). Once finished, the microworld filled the glass and the participants were asked to evaluate their solution. The second task was drawing a graph of filling the glass (see Figure 2). After seeing the microworld draw the correct graph on top of their own graph, the participants were asked to evaluate their solution.

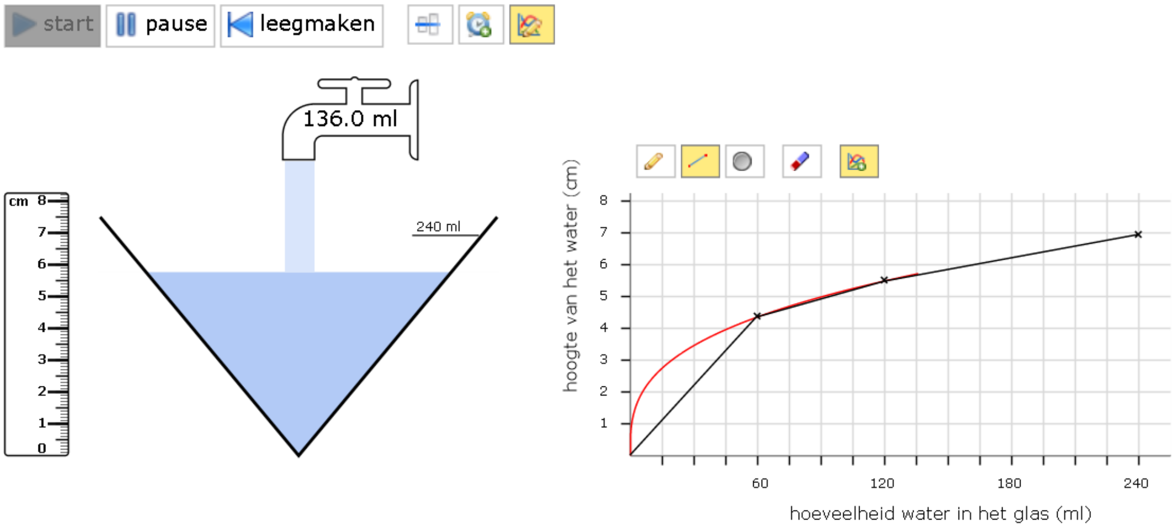

Figure 2: Evaluating drawing a graph of filling a cocktail glass in the FlaskFiller microworld: the red line is the graph drawn by the microworld whereas the black lines and crosses are user-drawn.

While performing the tasks, the participants were regularly invited to explain their reasoning. Although the evaluation phase after each task interferes with the think-aloud procedure, the aim of this experiment is to engage the participants in thinking about co-varying quantities and the more utterances the better for determination of the participant's level of reasoning.

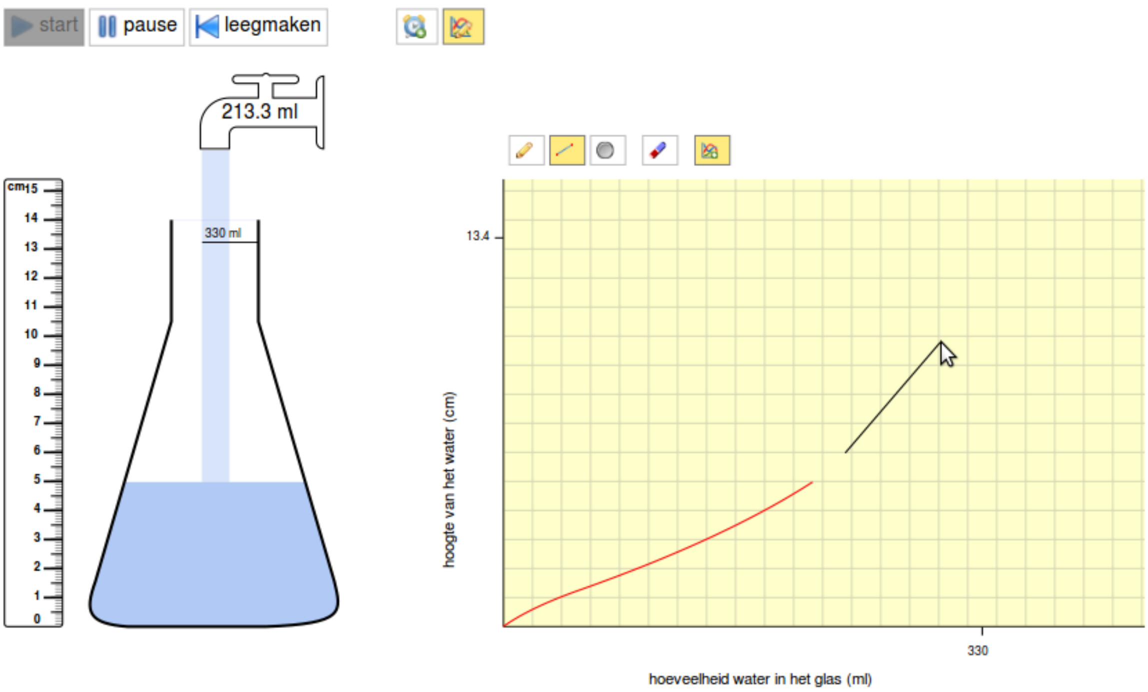

Figure 3: Drawing a graph of filling an Erlenmeyer flask in the FlaskFiller microworld without creating a measuring cup first.

During the pilot some participants constructed graphs based on the measuring cup by drawing line segments from one measure line to the next. It was unclear if this was an expression of the participant's understanding of change or just an application of a successful graph-drawing strategy. To prevent this behavior in subsequent experiments the third problem of filling an Erlenmeyer was adapted: instead of creating a measuring cup, the participant was asked to describe the graph of filling the Erlenmeyer followed by actual drawing the graph (see Figure 3).

4.4 Analysis

The videos of the experiments were transcribed and analyzed. All participants' explicit utterances about the co-variation of volume and water level were marked as an indication of one of five mental actions in the covariation framework. The screen capture videos were cut per task and the participant's performance during the whole task was classified as indicating behavior on one level of reasoning. The utterances and the task videos were also coded for pseudo-behavior.

The five developmental levels of reasoning about two co-varying quantities are: (1) coordination, (2) direction, (3) quantitative coordination, (4) average rate of change, and (5) instantaneous rate of change. The first level denotes the understanding that there is change involving volume and water level. The participants did not verbalize this basic understanding as it was implicit in the setting of the teaching experiment. At the second level one understands that the water level goes up when the volume increases. On the third level ones understanding attends to the amount of change of one quantity given increase of the other. Understanding at the fourth and fifth level focuses, respectively, on average rate of change of the water level given uniform intervals of the volume and the instantaneous rate of change of the water level given continuous increasing volume. The typical representational utterances indicating mental actions of these levels are presented in Table 1. In the following discussion the level of reasoning is put between parentheses.

level		measuring cup task	graph task
2	direction	Every next measure line is put above the previous measure line.	A straight line.
3	quantitative coordination	The distance between every two subsequent measure lines indicates the total amount of change of water level height given increases in volume between these measure lines.	The graph features certain points and/or line segments indicating the amount of change at these points or on these intervals.
4	average rate of change	if supported by verbal statements indicating average rate of change	A compound graph of line segments for uniform intervals of the volume.
5	instantaneous rate of change	not supported	A continuous curve.

Table 1: Typical behavior during the tasks per level of reasoning in the covariation framework. The first level is omitted as understanding on that level was implicit. The measuring cup task did not support behavior on levels four and five.

All participants could construct a measuring cup from the long drink glass (3). Creating a measuring cup from the cocktail glass, however, was more challenging as six participants tackled this problem exactly as the previous glass (2). After confrontation with filling the cocktail glass they all accepted that their linear solution was wrong and could explain why. During this evaluation phase their verbal statements were of a higher level of reasoning (3).

Finishing the partially drawn graph of filling a long drink glass was not a problem for the participants (3). Some participants just continued the line without regard for the situation (2). Drawing a graph of filling a cocktail glass was more difficult, except for the first participant (4). In the other seven experiments, four times the graphs drawn consisted of line segments with different slopes indicating that the glass filled up increasingly slower (3). In the previous task these participants created a linear measuring cup of the cocktail glass (2). In the other three experiments the participants just drew a straight line (2) indicating that they understood the general direction of change. In one of these experiments the measuring cup created was rated of a higher level of reasoning (3). When the participants were confronted with the curve drawn by the microworld they expressed their surprise for they did not know better than graphs are straight. Again, during this evaluation phase the participants' verbal statements were often of a higher level (3) than their task. In the last problem, except for the three experiments in the pilot, the participants drew either a straight line (2) or sketched some continuous curve indicating level (5) reasoning. As there was no further verbal or holistic support, these graphs were judged pseudo-behavior.

		Problem	
experiment	*long drink*	*cocktail*	*Erlenmeyer*
I (pilot)	3	4	4
II (pilot)	3	3	3
III (pilot, two participants)	2	3	3
IV	2	2	2
V	2	2	3
VI	2	3	3
VII	3	3	3
VIII	3	3	3

Table 2: The participants' level of reasoning in the covariation framework per problem.

The covariation framework can be used to analyze and determine a participant's level of reasoning about two co-varying quantities in one problem. In Table 2 the participants' level of reasoning per problem are given. In three of the experiments the level of reasoning seems to develop from (2) to (3); in four experiments the level of reasoning stays constant at (2) or (3); the participant in the first experiment is an excellent student and his level of reasoning exceed that of his peers (4). Although the problems were increasingly more difficult, the participants' level of reasoning over the whole experiment remained quite consistent.

4.5 Results

The participants reasoned at levels two and three of the covariation framework. To all participants it is obvious that the water level rises when the volume increases. They are quite capable estimating the relative amount of change at certain points or intervals. Except for the last problem, the participants almost never talk about rate of change and then it was pseudo behavior. Before the last problem the participants have seen the non-linear situation of the cocktail glass and its continuous curve. They know that the Erlenmeyer is similar and try to draw a curve. During the evaluation phase following, however, the participants are unable to characterize change beyond observations of what they see in the microworld. They observe the water level raising increasingly faster in the Erlenmeyer, but they probably associate this with dynamic aspects of the microworld; it is not an expression of their understanding of rate of change.

Once the participants broke through the linearity illusion in the cocktail glass problem, most participants reasoned at the quantitative coordination level (3). The participants' level of reasoning found in the study is quite consistent, suggesting it to be a reasonable indication of the average fifth graders' level of reasoning about similar situations of change. Using a suitable microworld 5th graders are able to understand change and express that understanding as the amount of change at certain points or intervals.

But how did the experiment and microworld help the participants to come to this level of reasoning? And how did their level of reasoning develop during the experiment? In class they will have solved a lot of partition problems similar to creating a measuring cup of a long drink glass, connecting this task to their previous knowledge. They lacked experience drawing graphs. Although some participants just continued the line without regard to the situation of change, the partially drawn graph did set them on the right track. This linear situation did not invite the participants to think explicitly about change, but it did fulfill its role as a training problem.

The next problem was more challenging. Without much thought six participants created a measuring cup of the cocktail glass by applying the same method as they did earlier. Only after seeing their measuring cup being filled, they started to notice a problem. It did not take long for the participants to understand why their solution was wrong. Through the instant feedback in the microworld the participants started reasoning about the co-varying quantities at a higher level. Their understanding of a non-linear situation of change stayed with them throughout the rest of the experiment.

In the graph task the participants either drew a straight line or constructed a graph from line segments. The participants were surprised by the curve drawn by the microworld, yet all agreed that the curve drawn by the microworld was correct. Understanding the amount of change at a point is fundamentally different (3) from understanding that the water level changes continuously (5). Drawing a curve expressing their understanding of change of filling an Erlenmeyer seemed a step too far.

The set-up of the teaching experiment helped the participants to get acquainted with the problem domain and the tasks through a simple and familiar problem. The switch to a non-linear situation of change forced the participants to critically look at their work. That enabled them to start thinking about non-linear patterns of change. Similarly, the confrontation with a continuous curve forced the participants to think about graphs and what they do represent. Some of the participants defaulted to one or more straight lines when they were unable to express their understanding of the situation as a curve, yet all knew it should be a curve.

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- Starting point: 5th graders reason about amount of change at certain points and intervals, not about rate of change
- Instant feedback helps breaking the linearity illusion
- A suitable microworld enables graphing for inexperienced graphers

5 Conclusion

In the previous sections a number of design principles for an instructional sequence on primary calculus were listed. Underlying most of these design principles is the idea that ICT enables younger students to tackle more complex problems by hiding more advanced concepts. This is a common aspect of many initiatives to teach calculus early. Another common characteristic, teaching average rate of change first, is less suitable for primary calculus. As an alternative to the standard mathematical definition of instantaneous rate of change we propose to use one closer to our intuitive notions about speed. Furthermore, the preference for the motion domain is detrimental to learn to understand change in a more general way.

By determining a reasonable indication of 5th graders' level of reasoning about change, starting points for an instructional sequence on primary calculus were found. When reasoning about situations of change in a non-motion context 5th graders do not reason about rate of change. They focus on the amount of change in certain points and are able to indicate the general direction of that change. 5th graders do need to break through the linearity illusion to start thinking about non-linear situations of change, but with instant feedback in a suitable microworld that did not appear to be a problem. Similarly, given their lack of graphing experience, 5th graders will be able to use graphs quite skillfully when supported by a suitable microworld. And that, in turn, enables teaching of primary calculus.

Notes

[1]Given function f modeling some situation of change. The instantaneous rate of change of f at moment m is given by: $lim_{h \to 0} \frac{f(m+h)-f(m)}{h}$. Here we compute the average rate of change on the interval *[*m,m+h*]*. As h is approaching *0*, the length of this interval approaches *0* and, as a result, the average rate of change on this interval approaches the instantaneous rate of change at moment m.

[2]This is easy to understand. Imagine a graph of a linear situation of change: a straight line. The average rate of change between every two points on a straight line is the same, no matter the size of the interval defined by those points, even if that interval approaches a size of *0* and approaches the instantaneous rate of change.

[3]See http://heerdebeer.org/flessenvuller/index.en.html for an English translation of the overview of the FlaskFiller microworld. At the top of the page you will find hyperlinks to the different glasses and bottles used in the teaching experiment.

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